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Deformations of bi-Hamiltonian structures of hydrodynamic type

Paolo Lorenzoni

International School for Advanced Studies (SISSA), Via Beirut 2-4, 34014 Trieste, Italy

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Abstract

In this paper we study the deformations of bi-Hamiltonian PDEs of hydrodynamic type with one dependent variable. The reason we study such deformations is that the deformed systems maintain an infinite number of commuting integrals of motion up to a certain order in the deformation parameter. This fact suggests that these systems could have, at least for small times, multi-soliton solutions. Our numerical experiments confirm this hypothesis.

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1. Introduction

The main purpose of this paper is to study the effects of a "deformation" of an infinite-dimensional completely integrable system.

The first point is to define what a deformation is. In this paper we deal with bi-Hamiltonian systems of hydrodynamic type for which it is quite natural to define deformations in terms of the Jacobi identity; the deformed bi-Hamiltonian structure satisfies the Jacobi identity only up to a certain order in the deformation parameter.

The interesting deformations are the deformations that cannot be obtained from the original bi-Hamiltonian structure by just a change of coordinates. Therefore, the first problem is to select the non-trivial deformations.

In order to solve this problem it is convenient to formulate it in terms of Poisson cohomology.

E-mail address: lorenzon@sissa.it (P. Lorenzoni).

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Recently, Degiovanni et al. [4] proved that the first two Poisson cohomology groups of a Poisson manifold (M, P) are trivial when M is the loop space $\{S^1 \to \mathbb{R}^n\}$ and P is a Poisson bracket of hydrodynamic type. Getzler [11] independently proved that all groups $H^i(P,M)$ for i positive are trivial for such (M,P).

This result, as we will see, simplifies remarkably our problem and allows us to solve it (in this paper we classify the deformations up to fourth order).

For the second-order deformations we show explicitly how to obtain an infinite "hierarchy" of Hamiltonian equations. We will see that the corresponding flows commute up to the order of the deformation.

One typical class of solutions of infinite-dimensional completely integrable systems is the class of multi-soliton solutions. A natural question arises: do the equations of the deformed hierarchy have multi-soliton solutions (at least for small times)?

The numerical experiments we have performed for an equation of the deformed hierarchy show the existence of solutions analogous to two-soliton solutions.

Finally, we observe that the deformations of the bi-Hamiltonian structures of hydrodynamic type appear in the framework of Frobenius manifolds where, with some additional constraints, they play a crucial role in the problem of reconstruction of a 2D TFT from a given Frobenius manifold studied by Dubrovin and Zhang [8]. One of these constraints, called quasi-triviality, is analyzed in the last part of this paper.

The paper is organized as follows. The first part (Section 2) is a brief introduction to Poisson cohomology. We focus our attention, in particular, on the infinite-dimensional version of Poisson cohomology. To do this we use the formalism of formal calculus of variations (see, e.g. [10]). The main purpose of Section 2 is to explain how to get the formulae for the Schouten brackets that will appear in the calculations. In Section 3 we give the classification of deformations up to fourth order in the deformation parameter. In Section 4 we construct the deformed hierarchy and we find one-soliton solutions of one equation of the hierarchy. Section 5 gives the results of the numerical experiments and Section 6 gives the proof of classification theorem. In the last section (Section 7) we introduce the notion of quasi-triviality and we prove that all deformations are quasi-trivial.

2. Poisson geometry

2.1. Poisson bracket

Definition 1. Let M be a smooth manifold, $P^{ij}(x)$ a bivector (i.e. a skew-symmetric contravariant tensor field of type (2, 0)), and f and g two smooth functions. The expression

$$\{f, g\} := P^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \tag{2.1}$$

is a Poisson bracket on M if it defines a structure of Lie algebra on the ring of smooth functions on M.

Jacobi identity is the only property of a Lie algebra that is not a consequence of the skew-symmetry of P^{ij} and of the definition of Poisson bracket.

It is well known that the Jacobi identity holds if and only if the tensor

$$J^{ijk} = \{\{x^i, x^j\}, x^k\} + \text{cyclic} = \frac{\partial P^{ij}}{\partial x^s} P^{sk} + \frac{\partial P^{jk}}{\partial x^s} P^{si} + \frac{\partial P^{ki}}{\partial x^s} P^{sj} \tag{2.2}$$

is indentically equal to 0. In fact

$$\{\{f, g\}, h\} + \text{cyclic} = J^{ijk} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \frac{\partial h}{\partial x^k} \tag{2.3}$$

for any f, g and $h \in C^\infty(M)$.

Definition 2. A vector field ∇H associated to a smooth function H by the formula

$$(\nabla H)^i = P^{ij} \frac{\partial H}{\partial x^j} = \{x^i, H\} \tag{2.4}$$

is called Hamiltonian vector field.

Definition 3. A smooth function f is called Casimir if $\{f, \cdot\} = 0$.

2.2. Schouten bracket and Poisson cohomology

Let $\Gamma^i(M)$ be the space of i-vectors. It is well known (see [6]) that there is a unique well defined R-bilinear extension of the Lie-derivative to an operator

$$[\cdot, \cdot] : \Gamma^p(M) \times \Gamma^q(M) \to \Gamma^{p+q}(M)$$

such that

$$[X_1 \wedge \cdots \wedge X_p, Q] = \sum_{i=1}^{p} (-1)^{i+1} X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_p \wedge [X_i, Q], \tag{2.5}$$

where $X_k \in \Gamma^1(M)$ for $k = 1, \ldots, p$, $Q \in \Gamma^q(M)$ and $[X_i, Q] = L_{X_i} Q$.

This bilinear map is called Schouten bracket. It has the following properties:

$$[P, Q] = (-1)^{pq} [Q, P], \tag{2.6}$$

$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{pq+q} Q \wedge [P, R], \tag{2.7}$$

$$(-1)^{p(r-1)} [P, [Q, R]] + (-1)^{q(p-1)} [Q, [R, P]] + (-1)^{r(q-1)} [R, [P, Q]] = 0, \tag{2.8}$$

where $P \in \Gamma^p(M)$, $Q \in \Gamma^q(M)$ and $R \in \Gamma^r(M)$.

The last property is called graded Jacobi identity. It can be proved by using (2.5) and (2.6).

In coordinates the Schouten bracket of a p-vector P and a q-vector Q is given by the formula (see [16]):

$$[P, Q]^{k_1 \cdots k_{p+q-1}} = \frac{(-1)^p}{p!(q-1)!} \delta^{k_1 \cdots k_{p+q-1}}_{i_1 \cdots i_p j_2 \cdots j_q} Q^{u j_2 \cdots j_q} \frac{\partial P^{i_1 \cdots i_p}}{\partial x^u} + \frac{1}{(p-1)!q!} \delta^{k_1 \cdots k_{p+q-1}}_{i_2 \cdots i_p j_1 \cdots j_q} P^{u i_2 \cdots i_p} \frac{\partial Q^{j_1 \cdots j_q}}{\partial x^u},$$

where $\delta^{\cdots}_{\cdots}$ is the Kronecker multi-index.

Example 1. If $P, Q \in \Gamma^2(M)$ then

$$[P, Q]^{ijk} = \frac{\partial Q^{ij}}{\partial x^s} P^{sk} + \frac{\partial P^{ij}}{\partial x^s} Q^{sk} + \text{cyclic}. \tag{2.9}$$

When $Q = P$ we obtain

$$\tfrac{1}{2}[P, P] = J^{ijk} \tag{2.10}$$

and then we have the following well known theorem.

Theorem 1. *The Jacobi identity holds if and only if* $[P, P] = 0$.

The last step before introducing the Poisson cohomolgy is the following theorem.

Theorem 2. *Let P be a Poisson bivector on M, then the operator* $d_P : \Gamma^q(M) \to \Gamma^{q+1}(M)$ *defined by the formula*:

$$d_P Q := [P, Q] \tag{2.11}$$

is a cohomology operator, i.e. $d_P^2 = 0$.

Proof. The graded Jacobi identity and the property (2.6) imply

$$0 = [[P, Q], P] + [[P, P], Q] + [[Q, P], P] = 2[[P, Q], P] = -2d_P^2 Q. \quad \square \tag{2.12}$$

The last theorem allows us, following [13], to introduce the complex

$$0 \to \Gamma^0(M) \to \Gamma^1(M) \to \Gamma^2(M) \to \Gamma^3(M) \cdots$$

and to define the Poisson cohomology as $HP^*(M, P) = \ker(d_P)/\mathrm{Im}(d_P)$.

Example 2 (HP^0). If f is a smooth function then $d_P f = P^{is}(\partial f/\partial x^s)$, i.e. $HP^0 =$ Casimirs.

Example 3 (HP^1). The cocycles are the infinitesimal symmetries ($d_P X = L_X P$), the coboundaries are Hamiltonian vector fields. Then $HP^1 =$ (infinitesimal symmetries)/(Hamiltonian vector fields).

Example 4 (HP^2). Let Q be a bivector. Q is a cocycle if and only if $[P, Q] = 0$. Therefore, $P + \epsilon Q$ satisfies Jacobi identity mod$(\mathrm{O}(\epsilon^2))$. This means that the cocycles are infinitesimal deformations of the Poisson bracket. The coboundaries are infinitesimal deformations obtained by a change of coordinates. In fact Q is a coboundary if and only if $Q = L_X P$ where X is a vector field. Summarizing $HP^2 =$ (infinitesimal deformations of P)/(deformations obtained by a change of coordinates).

2.3. Bi-Hamiltonian structure on M

Definition 4. A bi-Hamiltonian structure on M is a pair (P_1, P_2) of Poisson bivectors such that for any $\lambda_1, \lambda_2 \in R$, the linear combination

$$\lambda_1 P_1 + \lambda_2 P_2$$

is again a Poisson bivector.

In terms of Schouten bracket the bivectors (P_1, P_2) are a bi-Hamiltonian structure if and only if

$$[P_1, P_1] = [P_2, P_2] = [P_1, P_2] = 0. \tag{2.13}$$

2.4. Poisson brackets on formal loop space

Now we want to extend the previous definitions to the loop space $\mathcal{L} = \{u : S^1 \to \mathbb{R}\}$. Let $\mathcal{A}$ be the space of differential polynomials in u, i.e.:

$$f \in \mathcal{A} \Leftrightarrow f = \sum f_{s_1 \cdots s_m}(u) u^{(s_1)} \cdots u^{(s_m)}, \tag{2.14}$$

where $u^{(s_i)} := \mathrm{d}^{s_i} u / \mathrm{d}x^{s_i}$.

We observe that f is not necessarily a polynomial in u.

The role of the functions on $\mathcal{L}$ is played by the local functionals

$$I = \int_{S^1} f(u(x), u_x, u_{xx}, \dots)\, \mathrm{d}x,$$

where $f \in \mathcal{A}$.

Definition 5. A (non-local) multivector X is a formal infinite sum of the type

$$X = X^{s_1, \dots, s_k}(x_1, \dots, x_k; u(x_1), \dots, u(x_k), u_x(x_1), \dots) \frac{\partial}{\partial u^{s_1}(x_1)} \wedge \cdots \wedge \frac{\partial}{\partial u^{s_k}(x_k)},$$

where the coefficients satisfy the skew-symmetry condition with respect to simultaneous permutations

$$s_p, x_p \leftrightarrow s_q, x_q.$$

The wedge product of a k-vector X by an l-vector Y is defined as

$$(X \wedge Y)^{s_1, \dots, s_{k+l}}(x_1, \dots, x_{k+l}; u(x_1), \dots, u(x_{k+l}), \dots)$$

$$= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^{\operatorname{sgn} \sigma} X^{s_{\sigma(1)}, \dots, s_{\sigma(k)}}(x_{\sigma(1)}, \dots, x_{\sigma(k)}, \dots)$$

$$\times Y^{s_{\sigma(k+1)}, \dots, s_{\sigma(k+l)}}(x_{\sigma(k+1)}, \dots, x_{\sigma(k+l)}, \dots).$$

Definition 6. A k-vector is called translation invariant if

$$X^{s_1,\dots,s_k}(x_1,\dots,x_k;u(x_1),\dots,u(x_k),\dots) = \partial^{s_1}_{x_1}\cdots\partial^{s_k}_{x_k}X(x_1,\dots,x_k;u(x_1),\dots,u(x_k),\dots),$$

where $X(\cdot)$ means $X^{0\cdot 0}(\cdot)$ and for any t

$$X(x_1+t,\dots,x_k+t;u(x_1),\dots,u(x_k),\dots) = X(x_1,\dots,x_k;u(x_1),\dots,u(x_k),\dots).$$

It follows from this definition that a translation invariant multivector field is completely characterized by the "components"

$$X^{x_1\dots x_k} := X(x_1,\dots,x_k;u(x_1),\dots,u(x_k),\dots .).$$

Definition 7. A k form ω is a finite sum

$$\omega = \frac{1}{k!}\omega_{s_1\cdots s_k}\delta u^{s_1}\wedge\cdots\wedge\delta u^{s_k}, \tag{2.15}$$

where $\omega_{s_1\cdots s_k}\in\mathcal{A}$.

In order to define a Poisson structure we need to introduce a criterion for the Jacobi identity.

We have seen that in the finite-dimensional case the Jacobi identity can be written in terms of the Schouten bracket.

Therefore, if we will be able to define an infinite-dimensional version of the Schouten bracket we will be also able to define an infinite-dimensional version of the Poisson bracket.

2.4.1. Schouten bracket of translation invariant multivectors

In the case of translation invariant multivectors one obtains the formula for the Schouten bracket simply by translating the formula (2.9) in the new context. Heuristically this can be done by substituting sums for integrals, partial derivatives for variational derivatives, etc. (see [17]). The result is the following definition.

Definition 8. Schouten bracket of a translation invariant p-vector $P^{x_1\cdots x_p}$ and a translation invariant q-vector $Q^{x_1\cdots x_q}$:

$$\begin{aligned}[P,Q]^{x_1\cdots x_{p+q-1}} &= \sum_{\sigma\in S_{p+q-1}}(-1)^{\mathrm{sgn}(\sigma)}\sum_{s=0}\left(\sum_{i=1}^{p}\frac{(-1)^p}{p!(q-1)!}(\partial^s_{x_{\sigma(i)}}Q^{x_{\sigma(i)}x_{\sigma(p+1)}\cdots x_{\sigma(p+q-1)}})\right.\\ &\quad\times\left(\frac{\partial P^{x_{\sigma(1)}\cdots x_{\sigma(p)}}}{\partial u^{(s)}(x_{\sigma(i)})}\right)+\sum_{i=0}^{q-1}\frac{1}{(p-1)!q!}(\partial^s_{x_{\sigma(p+i)}}P^{x_{\sigma(p+i)}x_{\sigma(1)}\cdots x_{\sigma(p-1)}})\\ &\quad\left.\times\left(\frac{\partial Q^{x_{\sigma(p)}\cdots x_{\sigma(p+q-1)}}}{\partial u^{(s)}(x_{\sigma(p+i)})}\right)\right).\end{aligned} \tag{2.16}$$

In the case $p=2$

$$[P,Q]^{x_1\cdots x_{q+1}} = \sum_{\sigma\in S_{q+1}}(-1)^{\mathrm{sgn}(\sigma)}\sum_{s=0}\left(\sum_{i=1}^{p}\frac{1}{2!(q-1)!}(\partial^s_{x_{\sigma(i)}}Q^{x_{\sigma(i)}x_{\sigma(3)}\cdots x_{\sigma(q+1)}})\left(\frac{\partial P^{x_{\sigma(1)}x_{\sigma(2)}}}{\partial u^{(s)}(x_{\sigma(i)})}\right)\right.$$
$$\left.+\sum_{i=0}^{q-1}\frac{1}{q!}(\partial^s_{x_{\sigma(2+i)}}P^{x_{\sigma(2+i)}x_{\sigma(1)}})\left(\frac{\partial P^{x_{\sigma(2)}\cdots x_{\sigma(q+1)}}}{\partial u^{(s)}(x_{\sigma(2+i)})}\right)\right). \tag{2.17}$$

Example 5. Schouten bracket of a bivector P with the components P^{xy} and a local functional $I=\int_{S^1}f(u(x),u_x,u_{xx},\dots)\,\mathrm{d}x$

$$[P,I]^x = \int_{S^1}\frac{\delta f}{\delta u(y)}P_{yx}\,\mathrm{d}y. \tag{2.18}$$

Example 6. Lie-derivative of a translation invariant bivector P with the components P^{xy} along a translation invariant vector field Q with the components Q^x

$$[P,Q]^{xy} = \sum_s\left((\partial^s_x Q^x)\frac{\partial P^{xy}}{\partial u^s(x)}+(\partial^s_y Q^y)\frac{\partial P^{xy}}{\partial u^s(y)}+(\partial^s_y P^{yx})\frac{\partial Q^y}{\partial u^s(y)}\right.$$
$$\left.-(\partial^s_x P^{xy})\frac{\partial Q^x}{\partial u^s(x)}\right). \tag{2.19}$$

Example 7. Schouten bracket of two translation invariant bivectors P and Q:

$$[P,Q]^{xyz} = \frac{1}{2}\sum_s\left(\frac{\partial P^{xy}}{\partial u^s(x)}\partial^s_x Q^{xz}+\frac{\partial Q^{xy}}{\partial u^s(x)}\partial^s_x P^{xz}+\frac{\partial P^{xy}}{\partial u^s(y)}\partial^s_y Q^{yz}+\frac{\partial Q^{xy}}{\partial u^s(y)}\partial^s_y P^{yz}\right.$$
$$+\frac{\partial P^{zx}}{\partial u^s(z)}\partial^s_z Q^{zy}+\frac{\partial Q^{zx}}{\partial u^s(z)}\partial^s_z P^{zy}+\frac{\partial P^{zx}}{\partial u^s(x)}\partial^s_x Q^{xy}+\frac{\partial Q^{zx}}{\partial u^s(x)}\partial^s_x P^{xy}$$
$$\left.+\frac{\partial P^{yz}}{\partial u^s(y)}\partial^s_y Q^{yx}+\frac{\partial Q^{yz}}{\partial u^s(y)}\partial^s_y P^{yx}+\frac{\partial P^{yz}}{\partial u^s(z)}\partial^s_z Q^{zx}+\frac{\partial Q^{yz}}{\partial u^s(z)}\partial^s_z P^{zx}\right). \tag{2.20}$$

Remark 1. The operator $\partial/\partial u(x)$ is the usual partial derivative when it acts on functions depending on x while it is, by definition, equal to 0 when it acts on functions not depending on x. In other words

$$\frac{\partial u(x)}{\partial u(x)}=1,\qquad \frac{\partial u(y)}{\partial u(x)}=0.$$

Remark 2. The formula (2.19) can be obtained by the formula (2.17) for $q=1$ by using skew-symmetry of P.

Definition 9. A translation invariant Poisson bivector P is a translation invariant bivector such that

$$[P, P]_{\text{Schouten}} = 0.$$

As in finite-dimensional case a translation invariant Poisson bivector P^{xy} defines a Poisson structure on the loop space $\mathcal{L}$. The Poisson bracket of two local functionals I_1, I_2 is given by the formula:

$$\{I_1, I_2\} := \int_{S^1}\int_{S^1} \frac{\delta I_1}{\delta u(x)} P^{xy} \frac{\delta I_2}{\delta u(y)}\, \mathrm{d}x\, \mathrm{d}y. \tag{2.21}$$

To define the Poisson cohomology on the space of translation invariant multivectors we have to prove that the operator $d_P := [P, \cdot]$ associated to a Poisson bivector P satisfies the condition $d_P^2 = 0$. We have seen that this condition is satisfied (for a Poisson bivector) if the graded Jacobi identity holds and moreover that the graded Jacobi identity follows from (2.5) and (2.6).

The property (2.6) is obvious. As far as it concerns (2.5) it will be sufficient to analyze a particular case to understand how the proof works in general.

We want to check the "Leibniz rule" (2.5) when $p = 2$ and $q = 0$. In this case Q is a bivector. In this case we have to prove

$$[X \wedge Y, P] = Y \wedge [X, P] - X \wedge [Y, P]. \tag{2.22}$$

Left hand side

By using formula for the Schouten bracket of two bivector, we obtain

$$\begin{aligned}
&[X \wedge Y, P]_{xyz}\\
&= \left(\frac{\partial Y(y)}{\partial u^s(y)} X(x) - Y(x)\frac{\partial X(y)}{\partial u^s(y)}\right)(\partial_x^s P_{xz}) + \left(\frac{\partial P_{xy}}{\partial u^s(x)}\right)((\partial_x^s X)Y(z) - X(z)(\partial_x^s Y))\\
&\quad + \left(\frac{\partial Y(x)}{\partial u^s(x)} X(y) - Y(y)\frac{\partial X(x)}{\partial u^s(x)}\right)(\partial_y^s P_{yz}) + \left(\frac{\partial P_{xy}}{\partial u^s(x)}\right)((\partial_y^s X)Y(z) - X(z)(\partial_y^s Y))\\
&\quad + \left(\frac{\partial X(z)}{\partial u^s(z)} Y(x) - X(x)\frac{\partial Y(z)}{\partial u^s(z)}\right)(\partial_z^s P_{zy}) + \left(\frac{\partial P_{zx}}{\partial u^s(z)}\right)((\partial_z^s X)Y(y) - X(y)(\partial_z^s Y))\\
&\quad + \left(\frac{\partial Y(x)}{\partial u^s(x)} X(z) - Y(z)\frac{\partial X(x)}{\partial u^s(x)}\right)(\partial_x^s P_{xy}) + \left(\frac{\partial P_{zx}}{\partial u^s(x)}\right)((\partial_x^s X)Y(y) - X(y)(\partial_x^s Y))\\
&\quad + \left(\frac{\partial X(y)}{\partial u^s(y)} Y(z) - X(z)\frac{\partial Y(y)}{\partial u^s(y)}\right)(\partial_y^s P_{yx}) + \left(\frac{\partial P_{yz}}{\partial u^s(y)}\right)((\partial_y^s X)Y(x) - X(x)(\partial_y^s Y))\\
&\quad + \left(\frac{\partial Y(z)}{\partial u^s(z)} X(y) - Y(y)\frac{\partial X(z)}{\partial u^s(z)}\right)(\partial_z^s P_{zx}) + \left(\frac{\partial P_{yz}}{\partial u^s(z)}\right)((\partial_z^s X)Y(x) - X(x)(\partial_z^s Y)).
\end{aligned}$$

This formula can be written collecting terms in $Y(x)$, $Y(y)$, etc.:

$$\begin{aligned}
&= Y(x)\left(-\frac{\partial X(y)}{\partial u^s(y)}(\partial_y^s P_{yz}) + \frac{\partial X(z)}{\partial u^s(z)}(\partial_z^s P_{zy}) + \frac{\partial P_{yz}}{\partial u^s(y)}(\partial_y^s X(y))\right.\\
&\qquad \left. + \frac{\partial P_{yz}}{\partial u^s(z)}(\partial_z^s X(z))\right) + \cdots .
\end{aligned} \tag{2.23}$$

Right hand side

$$
\begin{aligned}
&Y \wedge [X, P] - X \wedge [Y, P] \\
&\quad = \frac{1}{2} \sum_{\sigma \in S_3} (-1)^{\mathrm{sgn}(\sigma)} Y(x_{\sigma(1)})[X, P]_{x_{\sigma(2)} x_{\sigma(3)}} \\
&\qquad - \frac{1}{2} \sum_{\sigma \in S_3} (-1)^{\mathrm{sgn}(\sigma)} X(x_{\sigma(1)})[Y, P]_{x_{\sigma(2)} x_{\sigma(3)}} \\
&\quad = \frac{1}{2}(Y(x)[X, P]_{yz} - Y(x)[X, P]_{zy} - Y(y)[X, P]_{xz} + Y(y)[X, P]_{zx} \\
&\qquad + Y(z)[X, P]_{xy} - Y(z)[X, P]_{yx}) - \frac{1}{2}(Y \leftrightarrow X).
\end{aligned}
$$

Let us consider the first two terms:

$$\tfrac{1}{2}(Y(x)[X, P]_{yz} - Y(x)[X, P]_{zy}).$$

They can be written as

$$
\begin{aligned}
Y(x) \Bigg(& \frac{\partial X(z)}{\partial u^s(z)} (\partial_z^s P_{zy}) - \frac{\partial X(y)}{\partial u^s(y)} (\partial_y^s P_{yz}) + \frac{1}{2} \frac{\partial P_{yz}}{\partial u^s(y)} (\partial_y^s X(y)) - \frac{1}{2} \frac{\partial P_{zy}}{\partial u^s(y)} (\partial_y^s X(y)) \\
& + \frac{1}{2} \frac{\partial P_{yz}}{\partial u^s(z)} (\partial_z^s X(z)) - \frac{1}{2} \frac{\partial P_{zy}}{\partial u^s(z)} (\partial_z^s X(z)) \Bigg).
\end{aligned}
$$

By comparing this expression with (2.23) we realize that they coincide because of the skew-symmetry:

$$\frac{\partial P_{zy}}{\partial u^s(y)} = -\frac{\partial P_{yz}}{\partial u^s(y)}, \tag{2.24}$$

$$\frac{\partial P_{zy}}{\partial u^s(z)} = -\frac{\partial P_{yz}}{\partial u^s(z)}. \tag{2.25}$$

Analogously for the other terms of the left and right hand side.

2.4.2. Local multivectors and their Poisson cohomology

Definition 10. A local k-vector is a translation invariant k-vector such that its dependence on $x_1, \dots, x_k$ is given by a finite order distribution with the support on the diagonal $x_1 = \cdots = x_k$.

In coordinates a multivector X has the form

$$X = \sum_{p_2, p_3, \dots, p_k \geq 0} X(u(x_1), u_x(x_1), \dots) \delta^{(p_2)}(x_1 - x_2) \cdots \delta^{(p_k)}(x_1 - x_k). \tag{2.26}$$

It is easy to check that $P_{xy} = u\delta'(x - y) + 1/2 u_x \delta(x - y)$ is a local bivector. In fact

$$
\begin{aligned}
P_{yx} &= u(y)\delta'(y - x) + \tfrac{1}{2} u_y \delta(y - x) \\
&= -u_x \delta'(x - y) - u_x \delta(x - y) + \tfrac{1}{2} u_x \delta(x - y) = -P_{xy}.
\end{aligned}
$$

But $\partial P_{xy}/\partial u(x)$ is not equal to $-\partial P_{yx}/\partial u(x) = 0$!

This problem can be solved by writing P_{xy} in a suitable form. If we write P_{xy} as $P'_{xy} = P_{xy} = 1/2(P_{xy} - P_{yx})$ then

$$\frac{\partial P'_{xy}}{\partial u(x)} = -\frac{\partial P'_{yx}}{\partial u(x)}. \tag{2.27}$$

This is true in general if we want to use the same formulae valid for non-local multivectors we have to write the local multivector in a form "compatible" with the operators $\partial/\partial u^s(x)$. The practical rule is to write the multivector $T^{x_1,\dots,x_n}$ in the form

$$\frac{1}{n!}\sum_{\sigma}(-1)^{\mathrm{sgn}(\sigma)}T^{x_{\sigma(1)},\dots,x_{\sigma(n)}}.$$

Analogously to the non-local case the formula (2.17) allows us to define a cohomology operator d_P starting from a local Poisson bivector P:

$$0 \to \Gamma^0 \xrightarrow{d_P} \Gamma^1 \xrightarrow{d_P} \Gamma^2_{\mathrm{local}} \xrightarrow{d_P} \Gamma^3_{\mathrm{local}} \to \cdots \xrightarrow{d_P} \Gamma^k_{\mathrm{local}} \cdots, \tag{2.28}$$

where Γ^0 is the space of local functionals, Γ^1 the space of non-local translation invariant vector fields and $\Gamma^k_{\mathrm{local}}$ the space of local translation invariant k-vector fields. It is easy to check that the cohomology groups we have introduced in this way have the same meaning that the cohomology groups of finite-dimensional Poisson manifolds (see the examples in the section devoted to Poisson cohomology of finite-dimensional Poisson manifolds).

3. Deformations of bi-Hamiltonian systems of hydrodynamic type

In this paper we deal with Poisson bivectors of the form

$$P^{xy} = \phi(u)\delta^{(1)}(x-y) + \frac{1}{2}(\partial_x\phi)\delta(x-y) + \sum_k \epsilon^k P^{[k]}_{xy}, \tag{3.1}$$

where

$$P^{[k]}_{xy} = \sum_{s=0}^{k+1} A_{k,s}(u, u_x, \dots)\delta^{(k+1-s)}(x-y), \tag{3.2}$$

where $A_{k,s}$ are differential polynomials. Introducing the following gradation in the space $\mathcal{A}$ of differential polynomials:

$$\deg(f(u)) = 0, \qquad \deg(u^{(k)}) = k.$$

We also require that

$$\deg(A_{k,s}) = s. \tag{3.3}$$

The part of order 0 is called local Poisson bracket of hydrodynamic type (see [6]).

Analogously to the finite-dimensional case we have the following definition.

Definition 11. A bi-Hamiltonian structure on the loop space L is a pair (P_1, P_2) of local Poisson bivectors such that for any $\lambda_1, \lambda_2 \in R$, the linear combination

$$\lambda_1 P_1 + \lambda_2 P_2$$

is again a Poisson bivector.

Definition 12. An evolutionary equation

$$u_t = F(u(x), u_x, u_{xx}, \dots.)$$

is called Hamiltonian if it can be written in the form

$$u_t = \{u, H\},$$

where H is some local functional.

Definition 13. An evolutionary equation is called bi-Hamiltonian if and only if it is Hamiltonian with respect to both Poisson bivectors of a bi-Hamiltonian structure.

Example 8. Rescaling $x \to \epsilon x$ the KdV equation $u_t = uu_x + u_{xxx}$ one obtains

$$u_t = \epsilon(uu_x + \epsilon^2 u_{xxx}).$$

One usually introduces slow time variable $t \to \epsilon t$ to rewrite the last equation into the form

$$u_t = uu_x + \epsilon^2 u_{xxx}.$$

This is small dispersion expansion of the KdV equation. It is a bi-Hamiltonian equation. The first Poisson bivector (Gardner–Zakharov–Faddeev bivector)

$$P^{xy} = \delta'(x - y) \tag{3.4}$$

and the second Poisson bivector (Magri bivector)

$$\{u(x), u(y)\}_2 = u\delta'(x - y) + \frac{1}{2}u_x\delta(x - y) - \epsilon^2\delta^3(x - y) \tag{3.5}$$

belong to the class of bivectors introduced before.

It is easy to check that the pair (P_1, P_2) of the Gardner–Zakharov–Faddeev bracket and the Magri bracket is a bi-Hamiltonian structure. It is called Magri bi-Hamiltonian structure.

Definition 14. The group of transformation

$$u \to \bar{u} = \sum_k \epsilon^k F_k(u, u_x, \dots), \tag{3.6}$$

where $F_k \in \mathcal{A}$, $\deg(F_k) = k$ and $\partial F_0/\partial u \neq 0$ is called Miura group.

Degiovanni et al. [4] and Getzler [11] solved independently the problem of studying the action of the Miura group on the bracket (3.2).

More precisely, they study the following problem: does it exits an element of the Miura group that transforms the bracket (3.2) into the bracket (3.4)?

This problem can be reduced to a cohomological problem. In fact we have seen that it corresponds to the study of the second group of Poisson cohomology associated to the bracket (3.4)

$HP^2(P, \Pi)$ = (infinitesimal deformations of P)/(deformations that can be obtained by infinitesimal change of coordinates of the form (3.6)).

This cohomology group is trivial (see [4,11]).

In this paper we deal with deformations of bi-Hamiltonian structure of hydrodynamic type.

Definition 15. A deformation of order k of a bi-Hamiltonian structure of hydrodynamic type is a pair (P_1, P_2) of the form (3.2) such that $P_1 - \lambda P_2$ satisfies Jacobi identity for every λ up to the order k:

$$[P_1, P_1] = [P_2, P_2] = [P_1, P_2] = \mathrm{o}(\epsilon^k). \tag{3.7}$$

From the triviality of second Poisson cohomology group it follows that we can always assume one of the brackets of the form (3.4).

In the next section we explain how to classify these deformations.

3.1. Classification of deformations of bi-Hamiltonian structure of hydrodynamic type

We are interested in the following question: which deformations of a bi-Hamiltonian structure of hydrodynamic type are trivial? In other words which deformations can be obtained from a bi-Hamiltonian structure of hydrodynamic type by the action of Miura group?

We start considering first-order deformations, i.e. $P_1 = P_1^{(0)} = \delta^{(1)}(x-y)$ and $P_2 = P_2^{(0)} + \epsilon P_2^{(1)}$.

By definition we have

$$[P_1, P_2] = \mathrm{o}(\epsilon), \tag{3.8}$$

$$[P_2, P_2] = \mathrm{o}(\epsilon). \tag{3.9}$$

The equation $[P_1, P_2] = \mathrm{o}(\epsilon)$ implies $[P_1^{(0)}, P_2^{(1)}] = d_1 P_2^{(1)} = 0$ and from the triviality of the second Poisson cohomology group it follows that there exists a vector $X_2^{(1)}$ such that

$$P_2^{(1)} = d_1 X_2^{(1)}. \tag{3.10}$$

The equation $[P_2, P_2] = \mathrm{o}(\epsilon)$ implies that

$$[P_2^{(0)}, P_2^{(1)}] = d_2 P_2^{(1)} = d_2 d_1 X_2^{(1)} = -d_1 d_2 X_2^{(1)} = 0, \tag{3.11}$$

where we have used the fact that $(d_1 + d_2)^2 = 0$.

Among all deformations that satisfy the equation $d_1 d_2 X_2^{(1)} = 0$ we have to select trivial deformations, i.e. the deformations that can be obtained by infinitesimal change of coordinates. In our case this means that there exists a vector field $\tilde{X}$ such that

$$\mathrm{Lie}_{\tilde{X}}\, P_1^{(0)} = 0, \tag{3.12}$$

$$\mathrm{Lie}_{\tilde{X}}\, P_2^{(0)} = P_2^{(1)}. \tag{3.13}$$

Theorem 3. *A deformation* $P_\lambda = P_1^{(0)} - \lambda(P_2^{(0)} + \epsilon d_1 X_2^{(1)})$ *is trivial* $\Leftrightarrow X_2^{(1)} = d_1 a + d_2 b$ *where a, b are local functionals.*

Proof. "$\Leftarrow$"

$$X_2^{(1)} = d_1 a + d_2 b \Rightarrow P_2^{(1)} = d_1 d_2 b = -d_2 d_1 b.$$

This means $P_2^{(1)} = \mathrm{Lie}_{\tilde{X}}\, P_2^{(0)}$ with $\tilde{X} = -d_1 b$.

Moreover $\mathrm{Lie}_{\tilde{X}}\, P_1^{(0)} = 0$.

"$\Rightarrow$" $\mathrm{Lie}_{\tilde{X}}\, P_1^{(0)} = 0 \Rightarrow \tilde{X} = d_1 b$ (the first Poisson cohomology group is trivial).

$$-d_1 d_2 b = d_2 d_1 b = \mathrm{Lie}_{\tilde{X}}\, P_2^{(0)} = P_2^{(2)} = d_1 X_2^{(2)} \Rightarrow X_2^{(2)} = -d_2 b + d_1 a. \qquad \square$$

From the last theorem it follows that the elements of the group $\mathrm{Ker}(d_1 d_2)/(\mathrm{Im}(d_1) + \mathrm{Im}(d_2))$ are the non-trivial first-order deformations.

Remark. For higher order deformations we can repeat the same arguments and obtain the same result. Consequently, in general:

1) A k-order deformation $P_2^{(k)}$ can be represented in the form

$$P_2^{(0)} - \lambda P_1^{(0)} + \sum_{i=1}^{k} \epsilon^i P_2^{(i)} = P_2^{(0)} - \lambda P_1^{(0)} + \sum_{i=1}^{k} \epsilon^i d_1 X_2^{(i)}, \tag{3.14}$$

where due to our definition of a gradation in the space $\mathcal{A}$ of differential polynomials (see (3.3)) we have necessarily $\deg(X_2^{(i)}) = i$ (see the form of the formula (2.19) when $P_1 = \delta'(x-y)$).

2) $P_2^{(k)}$ is trivial if and only if $X_2^{(k)} = d_1 A + d_2 B$.

Remark 3. Also in this case "trivial" means that $P_2^{(k)}$ can be eliminated by the action of Miura group. Obviously the triviality of $P_2^{(k)}$ does not imply that the k-order deformation is trivial.

Remark 4. From (2.18) it follows immediately that d_1 and d_2 increases the degree of a local functional by 1 (we will identify the degree of a local functional $\int f\,\mathrm{d}x$ with the degree of its density f). Then the degree of A and B must be equal to $k-1$.

In the next section we summarize the results of the classification of deformations up to fourth order.

3.2. Classification of deformations: results

Theorem 4. *Up to the fourth order all deformations of a bi-Hamiltonian structure of hydrodynamic type* (*see* Definition 15) *can be reduced, by the action of Miura group to the following form*

$$\begin{aligned} &u\delta^{(1)}(x-y)+\tfrac{1}{2}u_x\delta(x-y)-\lambda\delta^{(1)}(x-y)+\epsilon^2(-2s\delta^{(3)}(x-y)-3\partial_x s\delta^{(2)}(x-y)\\ &\quad-\partial_x^2 s\delta^{(1)}(x-y))+\epsilon^4(-2\tilde{s}\delta^{(5)}(x-y)-5(\partial_x\tilde{s})\delta^{(4)}(x-y)-10(\partial_x^2\tilde{s})\delta^{(3)}(x-y)\\ &\quad-10(\partial_x^3\tilde{s})\delta^{(2)}(x-y)-3(\partial_x^4\tilde{s})\delta^{(1)}(x-y)+2w\delta^{(3)}(x-y)+3(\partial_x w)\delta^{(2)}(x-y)\\ &\quad+(\partial_x^2 w)\delta^{(1)}(x-y))+\mathrm{O}(\epsilon^5) \end{aligned} \tag{3.15}$$

with

$$w=2\frac{\partial\tilde{s}}{\partial u}u_{xx}, \tag{3.16}$$

where s is an arbitrary function of u and $\tilde{s}=-2s(\partial s/\partial u)$.

From this theorem it follows immediately the following corollary.

Corollary 1. *Up to the fourth order every deformation of Magri bi-Hamiltonian structure is trivial.*

Proof. Indeed, in this case $s=$ constant $\neq 0$ and then $\tilde{s}=0$, $w=0$. □

4. The deformed hierarchy

4.1. Integrals of motions

As we have already been said bi-Hamiltonian structures give rise to an infinite number of “almost-constants” of motions. From these constants one can construct a hierarchy of Hamiltonian equations.

In this section we show explicitly how to produce this hierarchy and we check that the corresponding flows commute up to the order of the deformation.

The technique is well known for the KdV equation (see [9]); one looks for a solution of the equation

$$P_\lambda v=0 \tag{4.1}$$

in terms of a formal series. The coefficients of 1-form v are variational derivatives of the integrals of motions. In other words the 1-form v is exact with respect to the vertical differential of the variational bicomplex (see [3]).

We apply the same technique to the second-order deformations (see classification theorem).

We start with the equation $P_\lambda v=0$. This equation implies $vP_\lambda v=0$. From skew-symmetry of P_λ it follows that $0=vP_\lambda v=\partial_x(\cdot)$. That is $(\cdot)=f(\lambda)$. More precisely

$$v(-\lambda\partial_x+u\partial_x+\tfrac{1}{2}u_x+\epsilon^2(2s(u)\partial_x^3+3(\partial_x s)\partial_x^2+(\partial_x^2 s)\partial_x))v=0. \tag{4.2}$$

From (4.2) it follows

$$\partial_x\left(\tfrac{1}{2}v^2(u-\lambda)+\epsilon^2(2svv_{xx}+s_xvv_x-sv_x^2)\right)=0, \tag{4.3}$$

i.e.:

$$\tfrac{1}{2}v^2(u-\lambda)+\epsilon^2(2svv_{xx}+s_xvv_x-sv_x^2)=f(\lambda). \tag{4.4}$$

By choosing $f(\lambda)=-1/2\lambda$ we look for a solution of the form:

$$v=\sum_{i=0}\frac{p_i}{\lambda^i}, \tag{4.5}$$

where obviously $p_0=1$.

By straightforward calculation we get

$$\frac{1}{2}\sum_{i+j=k}\frac{p_ip_j}{\lambda^{k-1}}=\sum_{i+j=k-1}\left(\frac{u}{2}p_ip_j+\epsilon^2(2sp_ip_{jxx}+s_xp_ip_{jx}-sp_{ix}p_{jx}\right)\frac{1}{\lambda^{k-1}} \tag{4.6}$$

for $k=1$ (4.6) implies

$$\tfrac{1}{2}(p_0p_1+p_1p_0)=\tfrac{1}{2}up_0^2, \tag{4.7}$$

i.e.:

$$p_1=\tfrac{1}{2}u \tag{4.8}$$

for $k=2$

$$p_2=\tfrac{3}{8}u^2+\tfrac{1}{2}\epsilon^2(2su_{xx}+s_xu_x), \tag{4.9}$$

etc. We observe that the coefficients will have always the form

$$p_i=f(u)+\mathrm{O}(\epsilon^2)+\mathrm{O}(\epsilon^4)+\cdots, \tag{4.10}$$

where $f(u)$ is a polynomial in u.

Now we want to prove that the coefficients p_i are up to the second-order variational derivatives. We consider a curve $u(t)$ on the loop space L. By differentiating the equation

$$\tfrac{1}{2}v^2(u-\lambda)+\epsilon^2(2svv_{xx}+s_xvv_x-sv_x^2)+\tfrac{1}{2}\lambda=0 \tag{4.11}$$

along the vector field $\dot{u}$, tangent to this curve, we get

$$\begin{aligned}\dot{u}v&=-2\dot{v}(u-\lambda)-\epsilon^2\\&\quad\times\left(4\dot{s}v_{xx}+4s\frac{\dot{v}}{v}v_{xx}+4s\dot{v}_{xx}+2\dot{s}_xv_x+2s_x\frac{\dot{v}}{v}v_x+2s_x\dot{v}_x-2\dot{s}\frac{v_x^2}{v}-4s\frac{v_x}{v}\dot{v}_x\right).\end{aligned}$$

By straightforward calculation one can write the last equation in the form

$$\dot{u}v=\partial_t\left(-2\frac{\lambda}{v}\right)+\epsilon^2\left(\partial_t(-4sv_{xx}-2s_xv_x)+\partial_x\left(4s\frac{\dot{v}}{v}v_x\right)+2\frac{v_x}{v}(\dot{s}v_x-s_x\dot{v})\right). \tag{4.12}$$

This equation implies

$$\int_{S^1} \dot{u} v \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{S^1} f \, \mathrm{d}x\right) + \mathrm{O}(\epsilon^4) \tag{4.13}$$

for some $f = f(u, u_x, u_{xx}, \dots)$.

In fact from (4.10) it follows that v has the form

$$v = v^0 + \mathrm{O}(\epsilon^2), \tag{4.14}$$

where v^0 is a function of u and

$$\dot{s} v^0_x - s_x \dot{v}^0 = 0.$$

Now we can formulate the following theorem.

Theorem 5. *$p_i = \delta I_i/\delta u + \mathrm{O}(\epsilon^4)$ for some functionals I_i. Moreover*

$$\{I_i, I_j\}_1 = \mathrm{O}(\epsilon^4). \tag{4.15}$$

Proof.

$$\int_{S^1} \dot{u} v \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{S^1} f \, \mathrm{d}x\right) + \mathrm{O}(\epsilon^4)$$
$$= \int_{S^1} \left(\frac{\partial f}{\partial u}\dot{u} + \frac{\partial f}{\partial u_x}\dot{u}_x + \frac{\partial f}{\partial u_{xx}}\dot{u}_{xx} + \cdots\right) \mathrm{d}x + \mathrm{O}(\epsilon^4) = \cdots .$$

By integrating by parts, we get

$$\cdots = \int_{S^1} \dot{u} \left(\frac{\partial f}{\partial u} - \partial_x\left(\frac{\partial f}{\partial u_x}\right) + \partial_x^2\left(\frac{\partial f}{\partial u_{xx}}\right) + \cdots\right) \mathrm{d}x + \mathrm{O}(\epsilon^4)$$
$$= \int_{S^1} \dot{u} \frac{\delta f}{\delta u} \, \mathrm{d}x + \mathrm{O}(\epsilon^4).$$

Then

$$v = \frac{\delta f}{\delta u} + \mathrm{O}(\epsilon^4), \tag{4.16}$$

i.e.:

$$p_i = \frac{\delta I_i}{\delta u} + \mathrm{O}(\epsilon^4) \tag{4.17}$$

for some functionals I_i.

Moreover, by definition, the coefficients p_i of 1-form v satisfy the equation:

$$(P_2 - \lambda P_1)\left(\sum_{i=0} \frac{p_i}{\lambda^i}\right) = 0. \tag{4.18}$$

In terms of I_i this condition can be written as

$$\sum_{i=0}\left(P_2\frac{\delta I_i}{\delta u}\right)\frac{1}{\lambda^i}=\sum_{i=-1}\left(P_1\frac{\delta I_{i+1}}{\delta u}\right)\frac{1}{\lambda^i}+\mathrm{O}(\epsilon^4). \tag{4.19}$$

From this equation it follows immediately

$$P_1\frac{\delta I_0}{\delta u}=\mathrm{O}(\epsilon^4)$$
$$\cdots$$
$$P_2\frac{\delta I_i}{\delta u}=P_1\frac{\delta I_{i+1}}{\delta u}+\mathrm{O}(\epsilon^4)$$
$$\cdots.$$

By using these Lenard recursion relations it is easy to conclude the proof of the theorem (see for details [9]). □

4.2. *Soliton solutions*

Now we study one of the equation of the hierarchy. More precisely we concentrate on the equation

$$u_t=\partial_x\frac{\delta I_2}{\delta u}=\frac{3}{4}uu_x+\epsilon^2\left(s(u)u_{xxx}+2\frac{\partial s}{\partial u}u_xu_{xx}+\frac{1}{2}\frac{\partial^2 s}{\partial u^2}u_x^3\right). \tag{4.20}$$

When $s=$ constant this is KdV equation.

4.2.1. *One-soliton solutions*

We look for solutions of (4.20) of the form $u(x,t)=u(x+ct)=u(z)$.

By substituting we get

$$cu_z=\frac{3}{4}uu_z+\epsilon^2\left(s(u)u_{zzz}+2\frac{\partial s}{\partial u}u_zu_{zz}+\frac{1}{2}\frac{\partial^2 s}{\partial u^2}u_z^3\right). \tag{4.21}$$

This equation can be written as

$$\partial_z(cu-\tfrac{3}{8}u^2-\epsilon^2(\partial_z(su_z)-\tfrac{1}{2}(\partial_z s)u_z))=0, \tag{4.22}$$

i.e.:

$$cu-\tfrac{3}{8}u^2-\epsilon^2(\partial_z(su_z)-\tfrac{1}{2}(\partial_z s)u_z)=c_1. \tag{4.23}$$

By multiplying for u_z we get again a total derivative

$$\partial_z(\tfrac{1}{2}cu^2-\tfrac{1}{8}u^3-\epsilon^2(\tfrac{1}{2}su_z^2)-c_1u)=0, \tag{4.24}$$

i.e.:

$$\tfrac{1}{2}cu^2-\tfrac{1}{8}u^3-\epsilon^2(\tfrac{1}{2}su_z^2)-c_1u=c_2. \tag{4.25}$$

This equation can be written as

$$\left(\frac{\partial u}{\mathrm{d}z}\right)^2 = F(u), \tag{4.26}$$

where

$$F(u) = \frac{2}{\epsilon^2 s(u)}\left(-\frac{1}{8}u^3 + \frac{1}{2}cu^2 - c_1 u - c_2\right). \tag{4.27}$$

To obtain the solution one has to invert the following integral:

$$z - z_0 = \pm\int_u^{u_0} \frac{1}{F(u)^{1/2}}\,\mathrm{d}u. \tag{4.28}$$

4.2.2. Case $s(u) = u$

In this case $F(u) = P(u)/u$ where $P(u)$ is a polynomial of degree 3. It is well known (see [5]) that one-soliton solutions occur when $F(u)$ has one simple zero and one double zero.

By using the formula (see [15])

$$\int_x^{a_0}\left(\frac{x-a_3}{(a_0-x)(x-a_1)(x-a_2)}\right)^{1/2}\mathrm{d}x = \frac{2(a_0-a_3)}{((a_0-a_2)(a_1-a_3))^{1/2}}\Pi\left(\phi, \frac{a_1-a_0}{a_1-a_3}, k\right), \tag{4.29}$$

where $a_0 > u > a_1 > a_2 > a_3$ and

$$\phi = \arcsin\left(\frac{(a_1-a_3)(a_0-x)}{(a_0-a_1)(x-a_3)}\right)^{1/2}, \tag{4.30}$$

$$k = \left(\frac{(a_0-a_1)(a_2-a_3)}{(a_0-a_2)(a_1-a_3)}\right)^{1/2}, \tag{4.31}$$

$$\Pi(\phi, \nu, k) = \int_0^{\phi}\frac{\mathrm{d}\phi}{(1-\nu\sin^2\phi)(1-k^2\sin^2\phi)^{1/2}}, \tag{4.32}$$

we obtain, when $a_1 = a_2$ and $a_3 = 0$ (see [1]):

$$z - z_0 = \pm\frac{2a_1}{(a_1(a_0-a_1))^{1/2}}\left(-2\ln\left(\frac{(a_1(a_0-u)/u(a_0-a_1))^{1/2}+1}{(a_0(u-a_1)/u(a_0-a_1))^{1/2}}\right) - \left(\frac{a_0}{a_1}-1\right)^{1/2}\mathrm{arctg}\left(\frac{2u(a_0/u-1)^{1/2}}{2u-a_0}\right)\right). \tag{4.33}$$

The speed c of the wave and the constants of integration c_1 and c_2 can be easily written in terms of the coefficients a_0, a_1, a_2, a_3:

$$c = \tfrac{1}{4}(a_0 + 2a_1), \tag{4.34}$$

$$c_1 = \tfrac{1}{8}(2a_0a_1 + a_1^2), \tag{4.35}$$

$$c_2 = -\tfrac{1}{8}a_0a_1^2. \tag{4.36}$$

The expression (4.33) can be inverted numerically. To test the existence of two-soliton solutions we have used two such solutions with different speed as initial condition. The results of numerical experiments are described in the next section.

5. Numerical experiments

In this section we analyze Eq. (4.20) for $s(u) = u$, i.e.:

$$u_t = \partial_x\left(\frac{\delta I_2}{\delta u}\right) = \partial_x\left(\frac{3}{8}u^2 + \frac{\epsilon^2}{2}(2uu_{xx} + u_x^2)\right).$$

We write this equation in the form

$$u_t + F_x = 0,$$

where $F = -\delta I_2/\delta u$. To make numerical experiments we have used a two-step Lax–Wendroff scheme (see [14]). This scheme is characterized by an auxiliary step of calculation

$$u_{j+1/2}^{n+1/2} = \frac{1}{2}(u_j^n + u_{j+1}^n) - \frac{\Delta t}{2\Delta x}(F_{j+1}^n - F_j^n),$$

where $F_{j+1}^n = F(u_{j+1}^n)$.

The main step is

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(F_{j+1+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}).$$

To implement the two-step Lax–Wendroff scheme we have used a program in C language modified from [12]. Fig. 1 illustrates two solitons of different height and speed we have used in numerical experiments.

We have chosen periodic boundary conditions.

The results of our experiments is showed in Fig. 2. The waves move from the right to the left. The three pictures of the second figure show the solitons before, during and after the collision. Like the soliton collision in integrable systems they reemerge with the same shape.

Remark. Due to numerical instability for big amplitude we performed numerical experiments with small and slow waves.

We do not know if the origin of this numerical instability is the algorithm we have used.

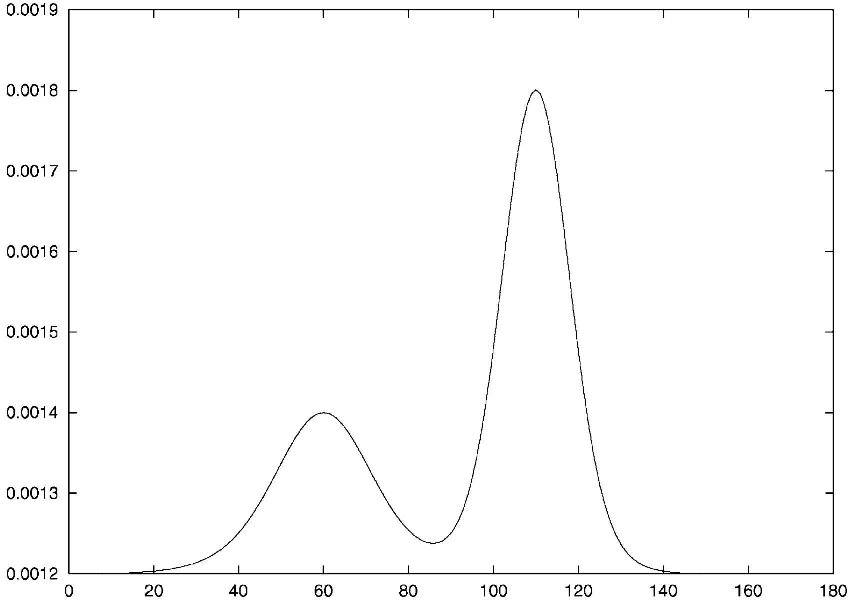

Fig. 1. Two solitons before the collision.

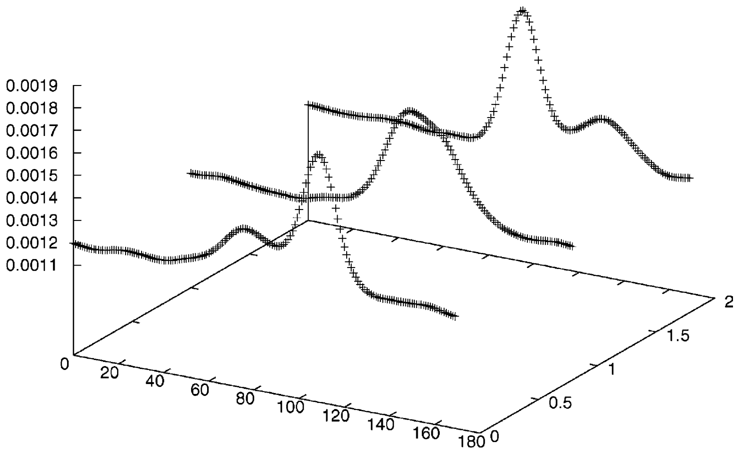

Fig. 2. Soliton collision.

6. Classification of deformations: proof

6.1. First-order deformations

We have seen that in this case the non-trivial deformations are the elements of the group $\mathrm{Ker}(d_1 d_2)/(\mathrm{Im}(d_1) + \mathrm{Im}(d_2))$.

We start looking for the solutions of the equation $d_1 d_2 X = 0$.

First of all we calculate $d_2 X$:

$$\begin{aligned} d_2X = [P_2^{(0)}, X]_{xy} &= \frac{1}{2}(\partial_x^s X)\frac{\partial}{\partial u^s(x)}(u\delta'(x-y) + \frac{1}{2}u_x\delta(x-y)) \\ &\quad - \frac{1}{2}(\partial_y^s X)\frac{\partial}{\partial u^s(y)}(u\partial_y\delta(y-x) + \frac{1}{2}u_y\delta(y-x)) + \partial_y^s(u\partial_y\delta(y-x) \\ &\quad + \frac{1}{2}u_y\delta(y-x))\frac{\partial X}{\partial u^s(y)} - \partial_x^s(u\delta'(x-y) + \frac{1}{2}u_x\delta(x-y))\frac{\partial X}{\partial u^s(x)}, \end{aligned}$$

where, by definition, $\delta'(x-y) = \delta^{(1)}(x-y) = \partial_x\delta(x-y)$.

Moreover by using the formula

$$f(y)\delta^{(s)}(x-y) = \sum_{q=0}^{s}\binom{s}{q} f^{(q)}(x)\delta^{(s-q)}(x-y), \tag{6.1}$$

we get

$$\begin{aligned} &\frac{1}{2}\left(\frac{\partial}{\partial u^s(x)}(u\delta'(x-y) + \frac{1}{2}u_x\delta(x-y))(\partial_x^s X) - \frac{\partial}{\partial u^s(y)}(u\partial_y\delta(y-x) \right. \\ &\quad \left. + \frac{1}{2}u_y\delta(y-x))(\partial_y^s X)\right) \\ &= \frac{1}{2}\left(X\delta'(x-y) + \frac{1}{2}\partial_x X\delta(x-y) - X\partial_y\delta(x-y) - \frac{1}{2}\partial_y X\delta(y-x)\right) \\ &= X\delta'(x-y) + \frac{1}{2}\partial_x X\delta(x-y) \end{aligned}$$

and

$$\begin{aligned} &\partial_y^s(u\partial_y\delta(y-x) + \frac{1}{2}u_y\delta(y-x))\frac{\partial X}{\partial u^s(y)} - \partial_x^s(u\delta^{(1)}(x-y) + \frac{1}{2}u_x\delta(x-y))\frac{\partial X}{\partial u^s(x)} \\ &= \partial_y^s(-u(x)\delta^{(1)}(x-y) - \frac{1}{2}u_x\delta(x-y))\frac{\partial X}{\partial u^s(y)} - \partial_x^s(-u(y)\partial_y\delta(y-x) \\ &\quad - \frac{1}{2}u_y\delta(y-x))\frac{\partial X}{\partial u^s(x)} \\ &= (-1)^{s+1}(u(x)\delta^{(s+1)}(x-y) + \frac{1}{2}u_x\delta^{(s)}(x-y))\frac{\partial X}{\partial u^s(y)} - u(y)\delta^{(s+1)}(x-y) \\ &\quad + \frac{1}{2}u_y\delta^{(s)}(x-y))\frac{\partial X}{\partial u^s(x)}. \end{aligned}$$

Summarizing we obtain

$$(d_2X)_{xy} = X\delta^{(1)}(x-y) + \frac{1}{2}\partial_x X\delta(x-y) + \sum_s\sum_{q=0}^{s+1} c_{qs}\delta^{(s+1-q)}(x-y) \tag{6.2}$$

with

$$c_{qs} = \binom{s+1}{q}\left((-1)^{s+1}u\left(\partial_x^q\left(\frac{\partial X}{\partial u^s(x)}\right)\right) - \frac{\partial X}{\partial u^s(x)}u^{(q)}\right) + \frac{1}{2}\binom{s}{q-1}\left((-1)^{s+1}u_x\left(\partial_x^{q-1}\left(\frac{\partial X}{\partial u^s(x)}\right)\right) + \frac{\partial X}{\partial u^s(x)}u^{(q)}\right). \tag{6.3}$$

In this case $\deg(X) = 1$, i.e. $X = s(u)u_x$. This implies that we can write the sum (6.2) as

$$(d_2X)_{xy} = (\tfrac{1}{2}\partial_x X + c_{21} + c_{10})\delta(x-y) + (X + c_{11} + c_{00})\delta^{(1)}(x-y) + c_{01}\delta^{(2)}(x-y) \tag{6.4}$$

with

$$c_{21} = \left(u\frac{\partial^2 s}{\partial u^2} + \frac{1}{2}\frac{\partial s}{\partial u}\right)u_x^2 + \left(u\frac{\partial s}{\partial u} - \frac{1}{2}s\right)u_{xx},$$

$$c_{10} = \left(-u\frac{\partial^2 s}{\partial u^2} - \frac{\partial s}{\partial u}\right)u_x^2 - u\frac{\partial s}{\partial u}u_{xx},$$

$$c_{11} = 2u\frac{\partial s}{\partial u}u_x - su_x, \qquad c_{00} = -2u\frac{\partial s}{\partial u}u_x, \qquad c_{01} = 0.$$

Substituting the last equations in (6.4) we obtain $(d_2X)_{xy} = 0$.

This means that all fields X of degree 1 belong to $\ker(d_1d_2)$.

Now we have to calculate the trivial field, i.e. the fields $X = d_1A + d_2B$, where

$$A = \int_{S^1} A(u)\,\mathrm{d}x, \qquad B = \int_{S^1} B(u)\,\mathrm{d}x.$$

Using the formulae

$$d_1A = -\partial_x\frac{\delta A}{\delta u} \tag{6.5}$$

and

$$d_2B = -\partial_x\left(u\frac{\delta B}{\delta u}\right) + \frac{1}{2}\frac{\delta B}{\delta u}u_x, \tag{6.6}$$

we get

$$d_1A + d_2B = \left(-\frac{\partial^2 A(u)}{\partial u^2} - \frac{1}{2}\frac{\partial B(u)}{\partial u} - u\frac{\partial^2 B(u)}{\partial u^2}\right)u_x. \tag{6.7}$$

This shows immediately that all deformations are trivial.

6.1.1. Explicit form of the deformations

We have to calculate d_1X for an arbitrary field of degree 1.

$d_2X = 0$ implies $d_1d_2X = -d_2d_1X = 0$. Then there exists a vector field $\tilde{X}$ of degree 1 such that $d_1X = d_2\tilde{X} = 0$.

This argument can be also used as an alternative proof of triviality of first-order deformations.

6.2. Second order

The triviality of first-order deformations means that we can always kill the term of first order in ϵ with a change of coordinates. Then we can reduce the second-order deformations to the form $P_2 = P_2^{(0)} - \lambda P_1^{(0)} + \epsilon^2 P_2^{(2)}$.

Also in this case the non-trivial deformations are elements of the group $\mathrm{Ker}(d_1 d_2)/(\mathrm{Im}(d_1) + \mathrm{Im}(d_2))$.

We start again considering the solutions of the equation $d_1 d_2 X = 0$.

In this case the sum (6.2) becomes

$$(d_2 X)_{xy} = (\tfrac{1}{2}\partial_x X + c_{10} + c_{21} + c_{32})\delta(x-y) + (X + c_{00} + c_{11} + c_{22})\delta^{(1)}(x-y) + (c_{01} + c_{12})\delta^{(2)}(x-y) + c_{02}\delta^{(3)}(x-y),$$

where $X = s_0 u_{xx} + s_1 u_x^2$ and

$$c_{10} = -\partial_x\left(u\frac{\partial X}{\partial u}\right), \quad c_{21} = u\partial_x^2\left(\frac{\partial X}{\partial u_x}\right) - \frac{1}{2}\frac{\partial X}{\partial u_x}u_{xx} + \frac{1}{2}\partial_x\left(\frac{\partial X}{\partial u_x}\right)u_x,$$

$$c_{32} = -u\partial_x^3\left(\frac{\partial X}{\partial u_{xx}}\right) - \frac{1}{2}\frac{\partial X}{\partial u_{xx}}u_{xxx} - \frac{1}{2}\partial_x^2\left(\frac{\partial X}{\partial u_{xx}}\right)u_x, \quad c_{00} = -2u\frac{\partial X}{\partial u},$$

$$c_{11} = 2u\partial_x\left(\frac{\partial X}{\partial u_x}\right) - \frac{\partial X}{\partial u_x}u_x, \quad c_{22} = -3u\partial_x^2\left(\frac{\partial X}{\partial u_{xx}}\right) - 2\frac{\partial X}{\partial u_{xx}}u_{xx} - \partial_x\left(\frac{\partial X}{\partial u_{xx}}\right)u_x,$$

$$c_{01} = 0, \quad c_{12} = -3u\partial_x\left(\frac{\partial X}{\partial u_{xx}}\right) - 3\frac{\partial X}{\partial u_{xx}}u_x, \quad c_{02} = -2u\frac{\partial X}{\partial u_{xx}}.$$

Hence we can write

$$(d_2 X)_{xy} = \sum_{k=0}^{3} b_k \delta^{(k)}(x-y) \tag{6.8}$$

with $\deg(b_k) = 3 - k$. By straightforward calculation we get

$$b_0 = au_{xxx} + bu_x u_{xx} + cu_x^3 = \left(-2u\frac{\partial s_0}{\partial u} + 2us_1\right)u_{xxx} + \left(s_1 - 4u\frac{\partial^2 s_0}{\partial u^2} + 4u\frac{\partial s_1}{\partial u} - \frac{\partial s_0}{\partial u}\right)u_x u_{xx} + \left(u\frac{\partial^2 s_1}{\partial u^2} + \frac{1}{2}\frac{\partial s_1}{\partial u} - u\frac{\partial^3 s_0}{\partial u^3} - \frac{1}{2}\frac{\partial s_0}{\partial u^2}\right)u_x^3,$$

$$b_1 = du_{xx} + gu_x^2 = \left(-5u\frac{\partial s_0}{\partial u} + 4us_1 - s_0\right)u_{xx} + \left(2u\frac{\partial s_1}{\partial u} - 3u\frac{\partial^2 s_0}{\partial u^2} - \frac{\partial s_0}{\partial u} - s_1\right)u_x^2,$$

$$b_2 = hu_x = -3\left(u\frac{\partial s_0}{\partial u} + s_0\right)u_x, \qquad b_3 = l = -2us_0.$$

Now we can calculate explicitly the equations $d_1d_2X = 0$ in terms of the coefficients b_0, b_1, b_2, b_3. By using the formula

$$
\begin{aligned}
(d_1d_2)_{xyz} = {} & \frac{1}{2}\sum_{t=0}^{3}\frac{\partial(d_2X)_{xy}}{\partial u^t(x)}\delta^{(t+1)}(x-z) - \frac{1}{2}\sum_{t=0}^{3}\frac{\partial(d_2X)_{yx}}{\partial u^t(y)}\delta^{(t+1)}(y-z) \\
& + \frac{1}{2}\sum_{t=0}^{3}\frac{\partial(d_2X)_{zx}}{\partial u^t(z)}\delta^{(t+1)}(z-y) - \frac{1}{2}\sum_{t=0}^{3}\frac{\partial(d_2X)_{xz}}{\partial u^t(x)}\delta^{(t+1)}(x-y) \\
& + \frac{1}{2}\sum_{t=0}^{3}\frac{\partial(d_2X)_{yz}}{\partial u^t(y)}\delta^{(t+1)}(y-x) - \frac{1}{2}\sum_{t=0}^{3}\frac{\partial(d_2X)_{zy}}{\partial u^t(z)}\delta^{(t+1)}(z-x),
\end{aligned}
$$

i.e.:

$$
\begin{aligned}
(d_1d_2)_{xyz} = {} & \frac{1}{2}\sum_{t=0}^{3}\left(\frac{\partial b_0}{\partial u^t(x)}\right)\delta(x-y)\delta^{(t+1)}(x-z) \\
& + \frac{1}{2}\sum_{t=0}^{2}\left(\frac{\partial b_1}{\partial u^t(x)}\right)\delta'(x-y)\delta^{(t+1)}(x-z) \\
& + \frac{1}{2}\sum_{t=0}^{1}\left(\frac{\partial b_2}{\partial u^t(x)}\right)\delta^{(2)}(x-y)\delta^{(t+1)}(x-z) \\
& + \frac{1}{2}\frac{\partial b_3}{\partial u(x)}\delta^{(3)}(x-y)\delta'(x-z) \\
& - \frac{1}{2}\sum_{t=0}^{3}\left(\frac{\partial b_0}{\partial u^t(y)}\right)\delta(y-x)\delta^{(t+1)}(y-z) \\
& - \frac{1}{2}\sum_{t=0}^{2}\left(\frac{\partial b_1}{\partial u^t(y)}\right)\delta'(y-x)\delta^{(t+1)}(y-z) \\
& - \frac{1}{2}\sum_{t=0}^{1}\left(\frac{\partial b_2}{\partial u^t(y)}\right)\delta^{(2)}(y-x)\delta^{(t+1)}(y-z) \\
& - \frac{1}{2}\frac{\partial b_3}{\partial u(y)}\delta^{(3)}(y-x)\delta'(y-z) + \cdots .
\end{aligned}
$$

In order to obtain a sum where only terms appear with $\delta^{(i)}(x-y)\delta^{(j)}(x-z)$ and where the coefficients depend only on x we use the identity

$$\delta(x-y)\delta(x-z) = \delta(y-x)\delta(y-z) = \delta(z-x)\delta(z-y) \tag{6.9}$$

and formula (6.1). For example we can write

$$-\frac{1}{2}\sum_{t=0}^{3}\left(\frac{\partial b_0}{\partial u^t(y)}\right)\delta(y-x)\delta^{(t+1)}(y-z)$$
$$= -(-1)^{t+1}\frac{1}{2}\sum_{t=0}^{3}\left(\frac{\partial b_0}{\partial u^t(y)}\right)\partial_z^{t+1}(\delta(y-x)\delta(y-z))$$
$$= -(-1)^{t+1}\frac{1}{2}\sum_{t=0}^{3}\left(\frac{\partial b_0}{\partial u^t(y)}\right)\partial_z^{t+1}(\delta(x-y)\delta(x-z))$$
$$= -\frac{1}{2}\sum_{t=0}^{3}\left(\frac{\partial b_0}{\partial u^t(y)}\right)\delta(x-y)\delta^{(t+1)}(x-z)$$
$$= -\frac{1}{2}\sum_{t=0}^{3}\left(\frac{\partial b_0}{\partial u^t(x)}\right)\delta(x-y)\delta^{(t+1)}(x-z).$$

In this way, after a long but elementary calculation we obtain

$$d_1 d_2 X = 2f\delta\wedge\delta^3 + 3f\delta^1\wedge\delta^2 + 3f_x\delta\wedge\delta^2 + f_{xx}\delta\wedge\delta^1, \tag{6.10}$$

where $\delta^i\wedge\delta^j = 1/2(\delta^{(i)}(x-y)\delta^{(j)}(x-z) - \delta^{(j)}(x-y)\delta^{(i)}(x-z))$ and

$$f = \frac{\partial b_1}{\partial u_x} - \partial_x\left(\frac{\partial b_1}{\partial u_{xx}}\right) - \frac{\partial b_2}{\partial u} + \partial_x\left(\frac{\partial b_3}{\partial u}\right). \tag{6.11}$$

Therefore, the equation $d_1 d_2 X = 0$ is equivalent to the equation $f = 0$. By substituting b_1, b_2, b_3 in this equation we get the condition

$$s_1 = \frac{\partial s_0}{\partial u}. \tag{6.12}$$

6.2.1. Trivial deformations

The differentials d_1 and d_2 increase the degree by 1. Because the degree of X is 2 the trivial deformations can be written as $X = d_1 A + d_2 B = 0$ where the degree of the densities of A and B is 1. But the variational derivative of local functionals with densities of degree 1 vanishes:

$$\frac{\delta\int_{S^1}(f(u)u_x)\,\mathrm{d}x}{\delta u} = \sum_{i=0}(-1)^i\partial_x^i\left(\frac{\partial f}{\partial u^{(i)}}\right) = \frac{\partial f}{\partial u}u_x - \partial_x(f(u)) = 0.$$

Then all the deformations $d_1 X_2^2$ (with $X_2^2 = s_0 u_{xx} + (\partial s_0/\partial u)u_x^2$) are not trivial.

6.2.2. Deformations: explicit form

By using the formula (2.19), we get

$$
\begin{aligned}
d_1X = \sum_s \Bigg(& (\partial_y^s\delta'(y-x))\frac{\partial}{\partial u^{(s)}(y)}\left(su_{yy}+\frac{\partial s}{\partial u}u_y^2\right) \\
& -(\partial_x^s\delta'(x-y))\frac{\partial}{\partial u^{(s)}(x)}\left(su_{xx}+\frac{\partial s}{\partial u}u_x^2\right)\Bigg) \\
= & \left(\frac{\partial s}{\partial u}u_{yy}+\frac{\partial^2 s}{\partial u^2}u_y^2\right)\delta^{(1)}(y-x) \\
& -\left(\frac{\partial s}{\partial u}u_{xx}+\frac{\partial^2 s}{\partial u^2}u_x^2\right)\delta^{(1)}(x-y) \\
& +\left(2\frac{\partial s}{\partial u}u_y\right)\delta^{(2)}(y-x)-\left(2\frac{\partial s}{\partial u}u_x\right)\delta^{(2)}(x-y) \\
& +s(y)\delta^{(3)}(y-x)-s(x)\delta^{(3)}(x-y).
\end{aligned}
$$

By observing that

$$
\frac{\partial s}{\partial u}u_{xx}+\frac{\partial^2 s}{\partial u^2}u_x^2=\partial_x^2 s
$$

and by using the identity (6.1) it is easy to get the formula

$$
P_2^{(2)}=d_1X_2^{(2)}=-2s\delta^3(x-y)-3\partial_x s\delta^2(x-y)-\partial_x^2 s\delta^1(x-y). \tag{6.13}
$$

6.3. Third order

The condition of compatibility $[P_1,P_2]=o(\epsilon^3)$ implies that $P_2^{(3)}=d_1X_2^{(3)}$ and the Jacobi identity $[P_2,P_2]=0$ implies that $d_2P_2^{(3)}=-d_1d_2X_2^{(3)}=o(\epsilon^3)$.

Remark. There are no conditions containing the second-order deformations.

We start again calculating d_2X. From (2.19) it follows:

$$
\begin{aligned}
(d_2X)_{xy} = & (\tfrac{1}{2}\partial_x X+c_{10}+c_{21}+c_{32}+c_{43})\delta(x-y) \\
& +(X+c_{00}+c_{11}+c_{22}+c_{33})\delta^{(1)}(x-y) \\
& +(c_{01}+c_{12}+c_{23})\delta^{(2)}(x-y) \\
& +(c_{02}+c_{13})\delta^{(3)}(x-y)+c_{03}\delta^{(4)}(x-y),
\end{aligned}
$$

where $X = s_0 u_{xxx} + s_1 u_x u_{xx} + s_2 u_x^3$ and

$$c_{10} = -\partial_x \left(u \frac{\partial X}{\partial u} \right), \qquad c_{21} = u \partial_x^2 \left(\frac{\partial X}{\partial u_x} \right) - \frac{1}{2} \frac{\partial X}{\partial u_x} u_{xx} + \frac{1}{2} u_x \partial_x \left(\frac{\partial X}{\partial u_x} \right),$$

$$c_{32} = -u \partial_x^3 \left(\frac{\partial X}{\partial u_{xx}} \right) - \frac{1}{2} \frac{\partial X}{\partial u_{xx}} u_{xxx} - \frac{1}{2} u_x \partial_x^2 \left(\frac{\partial X}{\partial u_{xx}} \right),$$

$$c_{43} = u \partial_x^4 \left(\frac{\partial X}{\partial u_{xxx}} \right) - \frac{1}{2} \frac{\partial X}{\partial u_{xxx}} u_{xxxx} + \frac{1}{2} u_x \partial_x^3 \left(\frac{\partial X}{\partial u_{xxx}} \right),$$

$$c_{00} = -2u \frac{\partial X}{\partial u}, \qquad c_{11} = 2u \partial_x \left(\frac{\partial X}{\partial u_x} \right) - \frac{\partial X}{\partial u_x} u_x,$$

$$c_{22} = -3u \partial_x^2 \left(\frac{\partial X}{\partial u_{xx}} \right) - 2 \frac{\partial X}{\partial u_{xx}} u_{xx} - u_x \partial_x \left(\frac{\partial X}{\partial u_{xx}} \right),$$

$$c_{33} = 4u \partial_x^3 \left(\frac{\partial X}{\partial u_{xxx}} \right) - \frac{5}{2} \frac{\partial X}{\partial u_{xxx}} u_{xxx} + \frac{3}{2} u_x \partial_x^2 \left(\frac{\partial X}{\partial u_{xxx}} \right),$$

$$c_{01} = 0, \qquad c_{12} = -3u \partial_x \left(\frac{\partial X}{\partial u_{xx}} \right) - 3 \frac{\partial X}{\partial u_{xx}} u_x,$$

$$c_{23} = 6u \partial_x^2 \left(\frac{\partial X}{\partial u_{xxx}} \right) - \frac{9}{2} \frac{\partial X}{\partial u_{xxx}} u_{xx} + \frac{3}{2} u_x \partial_x \left(\frac{\partial X}{\partial u_{xxx}} \right),$$

$$c_{13} = 4u \partial_x \left(\frac{\partial X}{\partial u_{xxx}} \right) - 3 \frac{\partial X}{\partial u_{xxx}} u_x, \qquad c_{02} = -2u \frac{\partial X}{\partial u_{xx}}, \qquad c_{03} = 0.$$

Then we can write

$$(d_2 X)_{xy} = \sum_{k=0}^{3} b_k \delta^{(k)}(x - y) \tag{6.14}$$

with $\deg(b_k) = 4 - k$ and

$$\begin{aligned} b_0 &= a u_{xxxx} + b u_x u_{xxx} + c (u_{xx})^2 + d u_x^2 u_{xx} + g u_x^4 \\ &= 0 + \left(-3u \frac{\partial s_1}{\partial u} + 3u \frac{\partial^2 s_0}{\partial u^2} + 6u s_2 \right) (u_x u_{xxx} + (u_{xx})^2) \\ &\quad + \left(-6u \frac{\partial^2 s_1}{\partial u^2} + 12u \frac{\partial s_2}{\partial u} - \frac{3}{2} \frac{\partial s_1}{\partial u} + 3 s_2 + 6u \frac{\partial^3 s_0}{\partial u^3} + \frac{3}{2} \frac{\partial^2 s_0}{\partial u^2} \right) u_x^2 u_{xx} \\ &\quad + \left(2u \frac{\partial^2 s_2}{\partial u^2} + \frac{\partial s_2}{\partial u} - \frac{1}{2} \frac{\partial^2 s_1}{\partial u^2} + u \frac{\partial^4 s_0}{\partial u^4} + \frac{1}{2} \frac{\partial^3 s_0}{\partial u^3} - u \frac{\partial^3 s_1}{\partial u^3} \right) u_x^4, \end{aligned}$$

$$
\begin{aligned}
b_1 &= hu_{xxx} + lu_x u_{xx} + mu_x^3\\
&= \left(2u\frac{\partial s_0}{\partial u} - us_1 - \frac{3}{2}s_0\right)u_{xxx}\\
&\quad + \left(12us_2 - 3s_1 - 9u\frac{\partial s_1}{\partial u} + 12u\frac{\partial^2 s_0}{\partial u^2} + \frac{3}{2}\frac{\partial s_0}{\partial u}\right)u_x u_{xx}\\
&\quad + \left(4u\frac{\partial s_2}{\partial u} - 2s_2 - 3u\frac{\partial^2 s_1}{\partial u^2} - \frac{\partial s_1}{\partial u} + 4u\frac{\partial^3 s_0}{\partial u^3} + \frac{3}{2}\frac{\partial^2 s_0}{\partial u^2}\right)u_x^3,\\
b_2 &= pu_{xx} + qu_x^2\\
&= \left(-3us_1 + 6u\frac{\partial s_0}{\partial u} - \frac{9}{2}s_0\right)u_{xx} + \left(-3u\frac{\partial s_1}{\partial u} - 3s_1 + 6u\frac{\partial^2 s_0}{\partial u^2} + \frac{3}{2}\frac{\partial s_0}{\partial u}\right)u_x^2,\\
b_3 &= ku_x = \left(-2us_1 + 4u\frac{\partial s_0}{\partial u} - 3s_0\right)u_x.
\end{aligned}
$$

6.3.1. Deformations

In terms of coefficients b_k the equation $(d_1 d_2 X)_{xy} = 0$ can be written simply by adding to the equation obtained in the previous case the terms

$$
\begin{aligned}
&\frac{1}{2}\left(\frac{\partial b_0}{\partial u_{xxxx}}\right)\delta(x-y)\delta^{(5)}(x-z) + \frac{1}{2}\left(\frac{\partial b_1}{\partial u_{xxx}}\right)\delta^{(1)}(x-y)\delta^{(4)}(x-z)\\
&\quad + \frac{1}{2}\left(\frac{\partial b_2}{\partial u_{xx}}\right)\delta^{(2)}(x-y)\delta^{(3)}(x-y) + \frac{1}{2}\left(\frac{\partial b_3}{\partial u_x}\right)\delta^{(3)}(x-y)\delta^{(2)}(x-z)\\
&\quad - \frac{1}{2}\left(\frac{\partial b_0}{\partial u_{yyyy}}\right)\delta(y-x)\delta^{(5)}(y-z) - \frac{1}{2}\left(\frac{\partial b_1}{\partial u_{yyy}}\right)\delta^{(1)}(y-x)\delta^{(4)}(y-z)\\
&\quad - \frac{1}{2}\left(\frac{\partial b_2}{\partial u_{yy}}\right)\delta^{(2)}(y-x)\delta^{(3)}(y-z) - \frac{1}{2}\left(\frac{\partial b_3}{\partial u_y}\right)\delta^{(3)}(y-x)\delta^{(2)}(y-z)\\
&\quad + \frac{1}{2}\left(\frac{\partial b_0}{\partial u_{zzzz}}\right)\delta(z-x)\delta^{(5)}(z-y) + \frac{1}{2}\left(\frac{\partial b_1}{\partial u_{zzz}}\right)\delta^{(1)}(z-x)\delta^{(4)}(z-y)\\
&\quad + \frac{1}{2}\left(\frac{\partial b_2}{\partial u_{zz}}\right)\delta^{(2)}(z-x)\delta^{(3)}(z-y) + \frac{1}{2}\left(\frac{\partial b_3}{\partial u_z}\right)\delta^{(3)}(z-x)\delta^{(2)}(z-y)\\
&\quad - \frac{1}{2}\left(\frac{\partial b_0}{\partial u_{xxxx}}\right)\delta(x-z)\delta^{(5)}(x-y) - \frac{1}{2}\left(\frac{\partial b_1}{\partial u_{xxx}}\right)\delta^{(1)}(x-z)\delta^{(4)}(x-y)\\
&\quad - \frac{1}{2}\left(\frac{\partial b_2}{\partial u_{xx}}\right)\delta^{(2)}(x-z)\delta^{(3)}(x-y) - \frac{1}{2}\left(\frac{\partial b_3}{\partial u_x}\right)\delta^{(3)}(x-z)\delta^{(2)}(x-y)\\
&\quad + \frac{1}{2}\left(\frac{\partial b_0}{\partial u_{yyyy}}\right)\delta(y-z)\delta^{(5)}(y-x) + \frac{1}{2}\left(\frac{\partial b_1}{\partial u_{yyy}}\right)\delta^{(1)}(y-z)\delta^{(4)}(y-x)
\end{aligned}
$$

$$+\frac{1}{2}\left(\frac{\partial b_2}{\partial u_{yy}}\right)\delta^{(2)}(y-z)\delta^{(3)}(y-x)+\frac{1}{2}\left(\frac{\partial b_3}{\partial u_y}\right)\delta^{(3)}(y-z)\delta^{(2)}(y-x)$$
$$-\frac{1}{2}\left(\frac{\partial b_0}{\partial u_{zzzz}}\right)\delta(z-y)\delta^{(5)}(z-x)-\frac{1}{2}\left(\frac{\partial b_1}{\partial u_{zzz}}\right)\delta^{(1)}(z-y)\delta^{(4)}(z-x)$$
$$-\frac{1}{2}\left(\frac{\partial b_2}{\partial u_{zz}}\right)\delta^{(2)}(z-y)\delta^{(3)}(z-x)+\frac{1}{2}\left(\frac{\partial b_3}{\partial u_z}\right)\delta^{(3)}(z-y)\delta^{(2)}(z-x).$$

By introducing the function f and g defined by the formulae

$$f=\frac{\partial b_1}{\partial u_x}-\partial_x\left(\frac{\partial b_1}{\partial u_{xx}}\right)+\partial_x^2\left(\frac{\partial b_1}{\partial u_{xxx}}\right)-\frac{\partial b_2}{\partial u}+\partial_x\left(\frac{\partial b_3}{\partial u}\right),$$
$$g=\frac{\partial b_1}{\partial u_{xxx}}-\frac{\partial b_2}{\partial u_{xx}}+\frac{\partial b_3}{\partial u_x},$$

we can write the result in the form

$$d_1d_2X=\sum f_{ij}\delta^i\wedge\delta^j, \tag{6.15}$$

where

$$f_{05}=f_{23}=2g,\qquad f_{14}=5g,\qquad f_{04}=5g_x,\qquad f_{13}=8g_x,$$
$$f_{03}=2f+4g_{xx},\qquad f_{12}=3f+3g_{xx},\qquad f_{02}=3f_x+g_{xxx},\qquad f_{01}=f_{xx}.$$

Therefore, the equation $d_1d_2X=0$ is equivalent to the equations $f=0$ and $g=0$.

By substituting the coefficients b_1, b_2 and b_3 in these equations we obtain that the last equation is identically satisfied and the first is equivalent to the condition

$$2s_2-\frac{\partial s_1}{\partial u}+\frac{\partial^2 s_0}{\partial u^2}=0. \tag{6.16}$$

6.3.2. Trivial deformations

In this case trivial deformations are d_1A+d_2B with

$$A=\int_{S^1}(A_0(u)u_{xx}+A_1(u)u_x^2)\,\mathrm{d}x,\qquad B=\int_{S^1}(B_0(u)u_{xx}+B_1(u)u_x^2)\,\mathrm{d}x.$$

By using formula (2.18) we get

$$d_1A+d_2B=\left(-2\left(\frac{\partial A_0}{\partial u}-A_1\right)-2u\left(\frac{\partial B_0}{\partial u}-B_1\right)\right)u_{xxx}$$
$$+\left(-4\left(\frac{\partial^2 A_0}{\partial u^2}-\frac{\partial A_1}{\partial u}\right)-4u\left(\frac{\partial^2 B_0}{\partial u^2}-\frac{\partial B_1}{\partial u}\right)-\left(\frac{\partial B_0}{\partial u}-B_1\right)\right)$$
$$+\left(-\left(\frac{\partial^3 A_0}{\partial u^3}-\frac{\partial^2 A_1}{\partial u^2}\right)-u\left(\frac{\partial^3 B_0}{\partial u^3}-\frac{\partial^2 B_1}{\partial u^2}\right)-\frac{1}{2}\left(\frac{\partial^2 B_0}{\partial u^2}-\frac{\partial B_1}{\partial u}\right)\right).$$

If we call $\tilde{A} := A_1 - \partial A_0/\partial u$ and $\tilde{B} := B_1 - \partial B_0/\partial u$, we can write

$$d_1 A + d_1 B = (2\tilde{A} + 2u\tilde{B})u_{xxx} + \left(4\frac{\partial \tilde{A}}{\partial u} + 4u\frac{\partial \tilde{B}}{\partial u} + \tilde{B}\right)u_x u_{xx} + \left(\frac{\partial^2 \tilde{A}}{\partial u^2} + 4u\frac{\partial^2 \tilde{B}}{\partial u^2} + \frac{\partial \tilde{B}}{\partial u}\right)u_x^3.$$

Now we can prove that all deformations $P_2^{(3)}$ are trivial.

Proof. The trivial deformations satisfy Eq. (6.16). In fact

$$2\left(\frac{\partial^2 \tilde{A}}{\partial u^2} + 4u\frac{\partial^2 \tilde{B}}{\partial u^2} + \frac{\partial \tilde{B}}{\partial u}\right) - \frac{\partial}{\partial u}\left(4\frac{\partial \tilde{A}}{\partial u} + 4u\frac{\partial \tilde{B}}{\partial u} + \tilde{B}\right) + \frac{\partial^2}{\partial u^2}(2\tilde{A} + 2u\tilde{B}) = 0.$$

Every field $X = s_0 u_{xxx} + s_1 u_x u_{xx} + s_2 u_x^3$ with coefficients s_0, s_1, s_2 satisfying Eq. (6.16) can be written as $d_1 A + d_2 B$ by choosing

$$\tilde{A} = \frac{s}{2} - \frac{u}{3}\left(2\frac{\partial s_0}{\partial u} - s_1\right), \qquad \tilde{B} = \frac{1}{3}\left(2\frac{\partial s_0}{\partial u} - s_1\right)$$

and this choice is always possible. □

6.3.3. Deformations: explicit form

By using the formula (2.19)

$$d_1 X = \sum_{s\geq 0}\left((\partial_y^s \delta'(y-x))\frac{\partial}{\partial u^{(s)}(y)}(s_0 u_{yyy} + s_1 u_y u_{yy} + s_2 u_y^3) - (\partial_x^s \delta'(x-y))\frac{\partial}{\partial u^{(s)}(x)}(s_0 u_{xxx} + s_1 u_x u_{xx} + s_2 u_x^3)\right)$$

and condition (6.16) it is easy to get the formula

$$P_2^{(3)} = d_1 X_2^{(3)} = -2t\delta^{(3)}(x-y) - 3\partial_x t\delta^{(2)}(x-y) - \partial_x^2 t\delta^{(1)}(x-y),$$

where t is an arbitrary differential polynomial of degree 1.

6.4. Fourth order

Due to the triviality of the third-order deformations, a fourth-order deformation can always be written in the form

$$P_2 = P_2^{(0)} + \epsilon^2 P_2^{(2)} + \epsilon^4 P_2^{(4)}, \tag{6.17}$$

where $P_2^{(2)} = d_1 X_2^{(2)}$ and $X_2^{(2)} = s u_{xx} + (\partial s/\partial u)u_x^2$.

The compatibility condition implies $P_2^{(4)} = d_1 X_2^{(4)}$ and the Jacobi identity $[P_2, P_2] = \mathrm{o}(\epsilon^4)$ implies

$$d_1 d_2 X_2^{(4)} - \tfrac{1}{2}[d_1 X_2^{(2)}, d_1 X_2^{(2)}] = 0.$$

6.4.1. Deformation

First of all we consider the term $[d_1 X_2^{(2)}, d_1 X_2^{(2)}]$. By using the formula (2.20) we get

$$\begin{aligned}
[d_1 X_2^{(2)}, d_1 X_2^{(2)}] = {} & \frac{\partial (d_1 X_2^{(2)})_{xy}}{\partial u^s(x)} \partial_x^s (d_1 X_2^{(2)})_{xz} - \frac{\partial (d_1 X_2^{(2)})_{yx}}{\partial u^s(y)} \partial_y^s (d_1 X_2^{(2)})_{yz} \\
& + \frac{\partial (d_1 X_2^{(2)})_{zx}}{\partial u^s(z)} \partial_z^s (d_1 X_2^{(2)})_{zy} - \frac{\partial (d_1 X_2^{(2)})_{xz}}{\partial u^s(x)} \partial_x^s (d_1 X_2^{(2)})_{xy} \\
& + \frac{\partial (d_1 X_2^{(2)})_{yz}}{\partial u^s(y)} \partial_y^s (d_1 X_2^{(2)})_{yx} - \frac{\partial (d_1 X_2^{(2)})_{zy}}{\partial u^s(z)} \partial_z^s (d_1 X_2^{(2)})_{zx}. \qquad (6.18)
\end{aligned}$$

Let us focus our attention on the first term. By straightforward calculation we get

$$\frac{\partial (d_1 X_2^{(2)})_{xy}}{\partial u^s(x)} \partial_x^s (d_1 X_2^{(2)})_{xz} = \sum_{i=1,\dots,3;\, j=1,\dots,6-i} b_{ij} \delta^{(i)}(x-y)\delta^{(j)}(x-z), \qquad (6.19)$$

where

$$\begin{aligned}
b_{11} = {} & \left(6 \frac{\partial^3 s}{\partial u^3} \frac{\partial^2 s}{\partial u^2} + 2 \frac{\partial s}{\partial u} \frac{\partial^4 s}{\partial u^4}\right) u_x^4 + \left(14 \left(\frac{\partial^2 s}{\partial u^2}\right)^2 + 14 \frac{\partial^3 s}{\partial u^3} \frac{\partial s}{\partial u}\right) u_x^2 u_{xx} \\
& + \left(8 \frac{\partial^2 s}{\partial u^2} \frac{\partial s}{\partial u}\right) u_{xx}^2 + \left(12 \frac{\partial^2 s}{\partial u^2} \frac{\partial s}{\partial u}\right) u_x u_{xxx} + 2 \left(\frac{\partial s}{\partial u}\right)^2 u_{xxxx},
\end{aligned}$$

$$b_{12} = \left(16 \frac{\partial^3 s}{\partial u^3} \frac{\partial s}{\partial u} + 16 \left(\frac{\partial^2 s}{\partial u^2}\right)^2\right) u_x^3 + \left(52 \frac{\partial^2 s}{\partial u^2} \frac{\partial s}{\partial u}\right) u_x u_{xx} + 10 \left(\frac{\partial s}{\partial u}\right)^2 u_{xxx},$$

$$b_{21} = \left(6 \frac{\partial^3 s}{\partial u^3} \frac{\partial s}{\partial u} + 6 \left(\frac{\partial^2 s}{\partial u^2}\right)^2\right) u_x^3 + \left(24 \frac{\partial^2 s}{\partial u^2} \frac{\partial s}{\partial u}\right) u_x u_{xx} + 6 \left(\frac{\partial s}{\partial u}\right)^2 u_{xxx},$$

$$b_{13} = \left(4 \frac{\partial^3 s}{\partial u^3} s + 38 \frac{\partial^2 s}{u^2} \frac{\partial s}{\partial u}\right) u_x^2 + \left(4 \frac{\partial^2 s}{\partial u^2} s + 18 \left(\frac{\partial s}{\partial u}\right)^2\right) u_{xx},$$

$$b_{31} = \left(4 \frac{\partial^2 s}{u^2} \frac{\partial s}{\partial u}\right) u_x^2 + 4 \left(\frac{\partial s}{\partial u}\right)^2 u_{xx}, \qquad b_{22} = \left(42 \frac{\partial^2 s}{u^2} \frac{\partial s}{\partial u}\right) u_x^2 + 24 \left(\frac{\partial s}{\partial u}\right)^2 u_{xx},$$

$$b_{23} = \left(12 \frac{\partial^2 s}{\partial u^2} s\right) u_x + 30 \left(\frac{\partial s}{\partial u}\right)^2, \qquad b_{32} = 12 \left(\frac{\partial s}{\partial u}\right)^2 u_x,$$

$$b_{14} = \left(8 \frac{\partial^2 s}{\partial u^2} s\right) u_x + 14 \left(\frac{\partial s}{\partial u}\right)^2, \qquad b_{24} = 12 \frac{\partial s}{\partial u} s,$$

$$b_{33} = 8 \frac{\partial s}{\partial u} s, \qquad b_{15} = 4 \frac{\partial s}{\partial u} s.$$

The other terms in (6.18) have the same form. The only difference is that the variables x, y and z play a different role. Therefore, we can apply the usual tricks and write an expression containing only terms with $\delta^{(i)}(x-y)\delta^{(j)}(x-z)$. For example we can write

$$\begin{aligned}
&-\frac{\partial(d_1X_2^{(2)})_{yx}}{\partial u^s(y)}\partial_y^s(d_1X_2^{(2)})_{yz}\\
&\quad=-\sum_{ij}b_{ij}(y)\delta^{(i)}(x-y)\delta^{(j)}(x-z)\\
&\quad=-(-1)^{i+j}\sum_{ij}b_{ij}(y)\partial_x^i\partial_z^j(\delta(y-x)\delta(y-z))\\
&\quad=-(-1)^{i+j}\sum_{ij}b_{ij}(y)\partial_x^i\partial_z^j(\delta(x-y)\delta(x-z))\\
&\quad=(-1)^{i+1}\sum_{ij}b_{ij}(y)\partial_x^i(\delta(x-y)\delta^{(j)}(x-z))\\
&\quad=(-1)^{i+1}\sum_{ij}b_{ij}(y)\sum_{k=0}^{i}\binom{i}{k}\delta^{(k)}(x-y)\delta^{(j+i-k)}(x-z)\\
&\quad=(-1)^{i+1}\sum_{ij}\sum_{k=0}^{i}\binom{i}{k}\sum_{l=0}^{k}\binom{k}{l}b_{ij}(x)^{(l)}\delta^{(k-l)}(x-y)\delta^{(j+i-k)}(x-z).
\end{aligned}$$

The final result is

$$\begin{aligned}
[d_1X_2^{(2)},d_1X_2^{(2)}]&=16\left(s\frac{\partial s}{\partial u}\right)\delta^1\wedge\delta^5+40\left(s\frac{\partial s}{\partial u}\right)\delta^2\wedge\delta^4\\
&\quad+40\partial_x\left(s\frac{\partial s}{\partial u}\right)\delta^1\wedge\delta^4+48\partial_x\left(s\frac{\partial s}{\partial u}\right)\delta^2\wedge\delta^3\\
&\quad+32\partial_x^2\left(s\frac{\partial s}{\partial u}\right)\delta^1\wedge\delta^3+8\partial_x^3\left(s\frac{\partial s}{\partial u}\right)\delta^1\wedge\delta^2.
\end{aligned}$$

The calculation of the term $d_2X_2^{(4)}$ can be done as above. In this case we have

$$\begin{aligned}
(d_2X_2^{(4)})_{xy}&=(\tfrac{1}{2}\partial_xX+c_{10}+c_{21}+c_{32}+c_{43}+c_{54})\delta(x-y)\\
&\quad\times(X+c_{00}+c_{11}+c_{22}+c_{33}+c_{44})\delta^{(1)}(x-y)\\
&\quad+(c_{01}+c_{12}+c_{23}+c_{34})\delta^{(2)}(x-y)+(c_{02}+c_{13}+c_{24})\delta^{(3)}(x-y)\\
&\quad+(c_{03}+c_{14})\delta^{(4)}(x-y)+c_{04}\delta^{(5)}(x-y),
\end{aligned}$$

where $X_2^{(4)}=X=s_0u_{xxxx}+s_1u_xu_{xxx}+s_2(u_{xx})^2+s_3u_x^2u_{xx}+s_4u_x^4$ and the coefficients c_{ij} have the same expression in terms of $X_2^{(4)}$ that in the previous case, except for the new coefficients

$$c_{54} = -u\partial_x^5\left(\frac{\partial X}{\partial u_{xxxx}}\right) - \frac{1}{2}\frac{\partial X}{\partial u_{xxxx}}u_{xxxxx} - \frac{1}{2}u_x\partial_x^4\left(\frac{\partial X}{\partial u_{xxxx}}\right),$$

$$c_{44} = -5u\partial_x^4\left(\frac{\partial X}{\partial u_{xxxx}}\right) - 3\frac{\partial X}{\partial u_{xxxx}}u_{xxxx} - 2u_x\partial_x^3\left(\frac{\partial X}{\partial u_{xxxx}}\right),$$

$$c_{34} = -10u\partial_x^3\left(\frac{\partial X}{\partial u_{xxxx}}\right) - 7\frac{\partial X}{\partial u_{xxxx}}u_{xxx} - 3u_x\partial_x^2\left(\frac{\partial X}{\partial u_{xxxx}}\right),$$

$$c_{24} = -10u\partial_x^2\left(\frac{\partial X}{\partial u_{xxxx}}\right) - 8\frac{\partial X}{\partial u_{xxxx}}u_{xx} - 2u_x\partial_x\left(\frac{\partial X}{\partial u_{xxxx}}\right),$$

$$c_{14} = -5u\partial_x\left(\frac{\partial X}{\partial u_{xxxx}}\right) - 5\frac{\partial X}{\partial u_{xxxx}}u_x, \qquad c_{04} = -2u\frac{\partial X}{\partial u_{xxxx}}.$$

Consequently we can write

$$(d_2X_2^{(4)})_{xy} = \sum_{k=0}^{5} b_k\delta^{(k)}(x-y)$$

with $\deg(b_k) = 5-k$ and

$$\begin{aligned}
b_0 &= a_0u_{xxxxx} + a_1u_xu_{xxxx} + a_2u_{xx}u_{xxx} + a_3u_x^2u_{xxx} + a_4u_x(u_{xx})^2 + a_5u_x^3u_{xx} + a_6u_x^5 \\
&= \left(2us_1 - 2us_2 - 2u\frac{\partial s_0}{\partial u}\right)u_{xxxxx} \\
&\quad + \left(s_1 - \frac{\partial s_0}{\partial u} - s_2 - 6u\frac{\partial s_2}{\partial u} + 6u\frac{\partial s_1}{\partial u} - 6u\frac{\partial^2 s_0}{\partial u^2}\right)u_xu_{xxxx} \\
&\quad + \left(-10u\frac{\partial s_2}{\partial u} + 10u\frac{\partial s_1}{\partial u} - 10u\frac{\partial^2 s_0}{\partial u^2}\right)u_{xx}u_{xxx} \\
&\quad + \left(2\frac{\partial s_1}{\partial u} - 2\frac{\partial s_2}{\partial u} - 2\frac{\partial^2 s_0}{\partial u^2} + 12us_4 - 4u\frac{\partial s_3}{\partial u} - 6u\frac{\partial^2 s_2}{\partial u^2} + 10u\frac{\partial^2 s_1}{\partial u^2}\right. \\
&\quad \left. - 10u\frac{\partial^3 s_0}{\partial u^3}\right)u_x^2u_{xxx} + \left(-\frac{3}{2}\frac{\partial s_2}{\partial u} + \frac{3}{2}\frac{\partial s_1}{\partial u} - \frac{3}{2}\frac{\partial^2 s_0}{\partial u^2} + 24us_4 - 8u\frac{\partial s_3}{\partial u} - 7u\frac{\partial^2 s_2}{\partial u^2}\right. \\
&\quad \left. + 15u\frac{\partial^2 s_1}{\partial u^2} - 15u\frac{\partial^3 s_0}{\partial u^3}\right)u_x(u_{xx})^2 + \left(6s_4 - 2\frac{\partial s_3}{\partial u} - \frac{\partial^2 s_2}{\partial u^2} + 3\frac{\partial^2 s_1}{\partial u^2} - 3\frac{\partial^3 s_0}{\partial u^3}\right. \\
&\quad \left. + 24u\frac{\partial s_4}{\partial u} - 8u\frac{\partial^2 s_3}{\partial u^2} - 2u\frac{\partial^3 s_2}{\partial u^3} + 10u\frac{\partial^3 s_1}{\partial u^3} - 10u\frac{\partial^4 s_0}{\partial u^4}\right)u_x^3u_{xxx} + \left(\frac{3}{2}\frac{\partial s_4}{\partial u}\right. \\
&\quad \left. - \frac{1}{2}\frac{\partial^2 s_3}{\partial u^2} + \frac{1}{2}\frac{\partial^3 s_1}{\partial u^3} - \frac{1}{2}\frac{\partial^4 s_0}{\partial u^4} + 3u\frac{\partial^2 s_4}{\partial u^2} - u\frac{\partial^3 s_3}{\partial u^3} + u\frac{\partial^4 s_1}{\partial u^4} - u\frac{\partial^5 s_0}{\partial u^5}\right)u_x^5,
\end{aligned}$$

$$
\begin{aligned}
b_1 &= c_0 u_{xxxx} + c_1 u_x u_{xxx} + c_2 (u_{xx})^2 + c_3 u_x^2 u_{xx} + c_4 u_x^4 \\
&= \left(-2s_0 - 7u\frac{\partial s_0}{\partial u} + 6us_1 - 6us_2\right) u_{xxxx} \\
&\quad + \left(-2s_2 - s_1 - 2\frac{\partial s_0}{\partial u} - 2us_3 - 12u\frac{\partial s_2}{\partial u} + 16u\frac{\partial s_1}{\partial u} - 20u\frac{\partial^2 s_0}{\partial u^2}\right) u_x u_{xxx} \\
&\quad + \left(-3s_2 - 2us_3 - 8u\frac{\partial s_2}{\partial u} + 12u\frac{\partial s_1}{\partial u} - 15u\frac{\partial^2 s_0}{\partial u^2}\right) (u_{xx})^2 \\
&\quad + \left(-5s_3 - 2\frac{\partial s_2}{\partial u} + \frac{9}{2}\frac{\partial s_1}{\partial u} - 6\frac{\partial^2 s_0}{\partial u^2} + 24us_4 - 13u\frac{\partial s_3}{\partial u} - 6u\frac{\partial^2 s_2}{\partial u^2}\right. \\
&\quad \left. + 24u\frac{\partial^2 s_1}{\partial u^2} - 30u\frac{\partial^3 s_0}{\partial u^3}\right) u_x^2 u_{xx} + \left(-3s_4 - \frac{\partial s_3}{\partial u} + \frac{3}{2}\frac{\partial^2 s_1}{\partial u^2} - 2\frac{\partial^3 s_0}{\partial u^3} + 6u\frac{\partial s_4}{\partial u}\right. \\
&\quad \left. - 3u\frac{\partial^2 s_3}{\partial u^2} + 4u\frac{\partial^3 s_1}{\partial u^3} - 5u\frac{\partial^4 s_0}{\partial u^4}\right) u_x^4, \\
b_2 &= d_0 u_{xxx} + d_1 u_x u_{xx} + d_2 u_x^3 \\
&= \left(-6us_2 + 6us_1 - 10u\frac{\partial s_0}{\partial u} - 7s_0\right) u_{xxx} \\
&\quad + \left(-6s_2 - 3s_1 - 3\frac{\partial s_0}{\partial u} - 6us_3 - 6u\frac{\partial s_2}{\partial u} + 18u\frac{\partial s_1}{\partial u} - 30u\frac{\partial^2 s_0}{\partial u^2}\right) u_x u_{xx} \\
&\quad + \left(-3s_3 + \frac{3}{2}\frac{\partial s_1}{\partial u} - 3\frac{\partial^2 s_0}{\partial u^2} - 3u\frac{\partial s_3}{\partial u} + 6u\frac{\partial^2 s_1}{\partial u^2} - 10u\frac{\partial^3 s_0}{\partial u^3}\right) u_x^3, \\
b_3 &= p_0 u_{xx} + p_1 u_x^2 \\
&= \left(-8s_0 - 4us_2 + 4us_1 - 10u\frac{\partial s_0}{\partial u}\right) u_{xx} \\
&\quad + \left(-3s_1 - 2\frac{\partial s_0}{\partial u} - 2us_3 + 4u\frac{\partial s_1}{\partial u} - 10u\frac{\partial^2 s_0}{\partial u^2}\right) u_x^2, \\
b_4 &= qu_x = \left(-5s_0 - 5u\frac{\partial s_0}{\partial u}\right) u_x, \qquad b_5 = k = -2us_0.
\end{aligned}
$$

We are almost able to write the equation

$$
d_1 d_2 X_2^{(4)} - \tfrac{1}{2}[d_1 X_2^{(2)}, d_1 X_2^{(2)}] = 0 \tag{6.20}
$$

in terms of b_k. In fact to obtain the term $d_1 d_2 X_2^{(4)}$ it is sufficient to add the following terms to the formula we have found before:

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial b_4}{\partial u}\delta^{(4)}(x-y)\delta^{(1)}(x-z) + \frac{1}{2}\frac{\partial b_5}{\partial u}\delta^{(5)}(x-y)\delta^{(1)}(x-z) \\
&\quad + \frac{1}{2}\frac{\partial b_4}{\partial u_x}\delta^{(4)}(x-y)\delta^{(2)}(x-z) + \frac{1}{2}\frac{\partial b_3}{\partial u_{xx}}\delta^{(3)}(x-y)\delta^{(3)}(x-z)
\end{aligned}
$$

$$+\frac{1}{2}\frac{\partial b_2}{\partial u_{xxx}}\delta^{(2)}(x-y)\delta^{(4)}(x-z)+\frac{1}{2}\frac{\partial b_1}{\partial u_{xxxx}}\delta^{(1)}(x-y)\delta^{(5)}(x-z)$$
$$+\frac{1}{2}\frac{\partial b_0}{\partial u_{xxxxx}}\delta(x-y)\delta^{(6)}(x-z)+\cdots$$

and to rewrite the terms in the usual way.

Defining the functions f, g, h in the following way

$$\begin{aligned}
f &:= \frac{\partial b_5}{\partial u}-\frac{\partial b_4}{\partial u_x}+\frac{\partial b_2}{\partial u_{xxx}}-\frac{\partial b_1}{\partial u_{xxxx}},\\
g &:= \frac{\partial b_1}{\partial u_{xxx}}-\frac{\partial b_2}{\partial u_{xxx}}+\frac{\partial b_3}{\partial u_x}-\frac{\partial b_4}{\partial u}-2\partial_x\left(\frac{\partial b_1}{\partial u_{xxxx}}\right)+\partial_x\left(\frac{\partial b_2}{\partial u_{xxx}}\right)\\
&\quad -\partial_x\left(\frac{\partial b_4}{\partial u_x}\right)+2\partial_x\left(\frac{\partial b_5}{\partial u}\right),\\
h &:= \frac{\partial b_1}{\partial u_x}-\partial_x\left(\frac{\partial b_1}{\partial u_{xx}}\right)+\partial_x^2\left(\frac{\partial b_1}{\partial u_{xxx}}\right)-\partial_x^3\left(\frac{\partial b_1}{\partial u_{xxxx}}\right)-\frac{\partial b_2}{\partial u}+\partial_x\left(\frac{\partial b_3}{\partial u}\right)\\
&\quad -\partial_x^2\left(\frac{\partial b_4}{\partial u}\right)+\partial_x^3\left(\frac{\partial b_5}{\partial u}\right),
\end{aligned}$$

we have

$$\begin{aligned}
d_1d_2X &= 2f\delta^1\wedge\delta^5+5f\delta^2\wedge\delta^4+2g\delta\wedge\delta^5+(5f_x+5g)\delta^1\wedge\delta^4\\
&\quad +(6f_x+2g)\delta^2\wedge\delta^3+5g_x\delta\wedge\delta^4+(4f_{xx}+8g_x)\delta^1\wedge\delta^3\\
&\quad +(2h+4g_{xx})\delta\wedge\delta^3+(f_{xxx}+3h\\
&\quad +3g_{xx})\delta^1\wedge\delta^2+(3h_x+g_{xxx})\delta\wedge\delta^2+h_{xx}\delta\wedge\delta^1.
\end{aligned}$$

Then the equation $d_1d_2X_2^{(4)}-1/2[d_1X_2^{(2)},d_1X_2^{(2)}]=0$ is equivalent to the conditions

$$f-4s\frac{\partial s}{\partial u}=0,\qquad g=0,\qquad h=0.$$

The first equation connects the second-order deformation with the fourth-order deformation:

$$s_0=-2s\frac{\partial s}{\partial u}. \tag{6.21}$$

The second and the third equation give the conditions

$$s_1-s_2=\frac{\partial s_0}{\partial u}, \tag{6.22}$$

$$3s_4-\frac{\partial s_3}{\partial u}+\frac{\partial^2 s_2}{\partial u^2}=0. \tag{6.23}$$

6.4.2. Trivial deformations

In this case trivial deformations can be written as d_1A+d_2B with

$$A=\int_{S^1}(A_0(u)u_{xxx}+A_1(u)u_xu_{xx}+A_2(u)u_x^3)\,\mathrm{d}x, \tag{6.24}$$

$$B=\int_{S^1}(B_0(u)u_{xxx}+B_1(u)u_xu_{xx}+B_2(u)u_x^3)\,\mathrm{d}x. \tag{6.25}$$

A straightforward calculation shows that, for trivial deformations, the coefficients s_i are

$$s_0=0, \tag{6.26}$$

$$s_1=s_2=3\tilde{A}+3u\tilde{B}, \tag{6.27}$$

$$s_3=6\frac{\partial\tilde{A}}{\partial u}+6u\frac{\partial\tilde{B}}{\partial u}+\frac{3}{2}\tilde{B}, \tag{6.28}$$

$$s_4=\frac{\partial^2\tilde{A}}{\partial u^2}+6u\frac{\partial^2\tilde{B}}{\partial u^2}+\frac{1}{2}\frac{\partial\tilde{B}}{\partial u} \tag{6.29}$$

with

$$\tilde{A}=2A_2-\frac{\partial A_1}{\partial u}+\frac{\partial^2 A_0}{\partial u^2},\qquad \tilde{B}=2B_2-\frac{\partial B_1}{\partial u}+\frac{\partial^2 B_0}{\partial u^2}.$$

Now we are able to prove that $P_2^{(4)}$ is trivial if and only if $s_0=0$.

Proof. Trivial deformations satisfy equations

$$s_0=0,\qquad s_1=s_2,\qquad 3s_4-\frac{\partial s_3}{\partial u}+\frac{\partial^2 s_2}{\partial u^2}=0.$$

In fact

$$3\frac{\partial^2}{\partial u^2}\left(\tilde{A}+\tilde{B}\right)-\frac{3}{2}\frac{\partial\tilde{B}}{\partial u}-\frac{\partial}{\partial u}\left(6\frac{\partial}{\partial u}(\tilde{A}+\tilde{B})-\frac{9}{2}\tilde{B}\right)+\frac{\partial^2}{\partial u^2}(3(\tilde{A}+\tilde{B}))=0.$$

Conversely, we can always write a deformation X (with $s=0$) as $X=d_1A+d_2B$. It is sufficient to choose A and B in such way that the following equations hold:

$$\tilde{B}=\frac{4}{9}\frac{\partial s_2}{\partial u}-\frac{2}{9}s_3,\qquad \tilde{A}=\frac{1}{3}s_2-\tilde{B}.$$ □

6.4.3. Deformations: explicit form

By using the formula (2.19)

$$d_1X=\sum_s\left((\partial_y^s\delta'(y-x))\frac{\partial}{\partial u^{(s)}(y)}(s_0u_{yyyy}+s_1u_yu_{yyy}+s_2u_{yy}^2+s_3u_y^4)\right.$$
$$\left.-(\partial_x^s\delta'(x-y))\frac{\partial}{\partial u^{(s)}(x)}(s_0u_{xxxx}+s_1u_xu_{xxx}+s_2u_{xx}^2+s_3u_x^4)\right)$$

and conditions (6.21)–(6.23), it is easy to get the formula

$$\begin{aligned}d_1X_2^{(4)}&=-2s_0\delta^{(5)}(x-y)-5(\partial_xs_0)\delta^{(4)}(x-y)-10(\partial_x^2s_0)\delta^{(3)}(x-y)\\&\quad-10(\partial_x^3s_0)\delta^{(2)}(x-y)-3(\partial_x^4s_0)\delta^{(1)}(x-y)+2w\delta^{(3)}(x-y)\\&\quad+3(\partial_xw)\delta^{(2)}(x-y)+(\partial_x^2w)\delta^{(1)}(x-y)\end{aligned}$$

with

$$w = w_0 u_{xx} + w_1 u_x^2 = 2\frac{\partial s_0}{\partial u} u_{xx} + w_1 u_x^2, \tag{6.30}$$

where w_1 is an arbitrary function of u and s_0 is related to the function s appearing in second-order deformation by the equation

$$s_0 = -2s\frac{\partial s}{\partial u}. \tag{6.31}$$

7. Quasi-triviality

Definition 16. The group of transformations

$$u \to \bar{u} = \sum_k \epsilon^k \frac{F_k(u, u_x, u_{xx}, \dots)}{G_k(u, u_x, u_{xx}, \dots)}, \tag{7.1}$$

where $F_k, G_k \in A$, $\deg(F_k) - \deg(G_k) = k$ and $\partial F_0/\partial u \neq 0$ is called quasi-Miura group.

A deformation is called quasi-trivial if it can be eliminated by the action of the quasi-Miura group.

We have seen that, in general, the second- and fourth-order deformations are not trivial. In this section we show that they are quasi-trivial.

7.1. Second order

Theorem 6. *If $X_2^{(2)} = d_1 A + d_2 B$, where A and B are local functionals whose densities are ratios of differential polynomials, then the second-order deformation $P_\lambda = P_1^{(0)} - \lambda(P_2^{(0)} + \epsilon^2 d_1 X_2^{(2)})$ is trivial.*

Proof.

$$X_2^{(2)} = d_1 A + d_2 B \Rightarrow P_2^{(2)} = -d_2 d_1 B.$$

This implies

$$P_2^{(2)} = \mathrm{Lie}_X\, P_2^{(0)}, \qquad \mathrm{Lie}_X\, P_1^{(0)} = 0$$

with

$$X = -d_1 B. \qquad \square \tag{7.2}$$

Theorem 7. *All second-order deformations are quasi-trivial.*

Proof. If we choose

$$A = \int_{S^1} \left(a(u)\frac{u_{xx}}{u_x} \right) \mathrm{d}x, \tag{7.3}$$

$$B = \int_{S^1} \left(b(u) \frac{u_{xx}}{u_x} \right) \mathrm{d}x, \tag{7.4}$$

by using formula (2.18) we get

$$d_1 A + d_2 B = \left(2 \frac{\partial^2 a}{\partial u^2} + 2u \frac{\partial^2 b}{\partial u^2} + \frac{1}{2} \frac{\partial b}{\partial u} \right) u_{xx} - \left(\frac{\partial a}{\partial u} + u \frac{\partial b}{\partial u} \right) \frac{u_{xx}^2}{u_x^2} + \left(\frac{\partial a}{\partial u} + u \frac{\partial b}{\partial u} \right) \frac{u_{xxx}}{u_x} + \left(\frac{\partial^3 a}{\partial u^3} + u \frac{\partial^3 b}{\partial u^3} + \frac{1}{2} \frac{\partial^2 b}{\partial u^2} \right) u_x^2.$$

If we put

$$a(u) = -ub(u) + \int_{u_0}^{u} b(u)\, \mathrm{d}u, \tag{7.5}$$

then

$$\frac{\partial a}{\partial u} = -u \frac{\partial b}{\partial u}, \qquad \frac{\partial^2 a}{\partial u^2} = -\frac{\partial b}{\partial u} - u \frac{\partial^2 b}{\partial u^2}, \qquad \frac{\partial^3 a}{\partial u^3} = -2 \frac{\partial^2 b}{\partial u^2} - u \frac{\partial^3 b}{\partial u^3}$$

and these equations imply

$$d_1 A + d_2 B = -\frac{3}{2} \left(\frac{\partial b}{\partial u} u_{xx} + \frac{\partial^2 b}{\partial u^2} u_x^2 \right) \tag{7.6}$$

that is equal to $X_2^{(2)} = s(u) u_{xx} + (\partial s / \partial u) u_x^2$ if we choose

$$b(u) = -\frac{2}{3} \int_{u_0}^{u} s(u)\, \mathrm{d}u. \qquad \square \tag{7.7}$$

7.2. Fourth order

Lemma 1. *A fourth-order deformation is quasi-trivial if and only if there exists a vector field Y such that*

$$\mathrm{Lie}_Y\, P_1^{(0)} = 0, \tag{7.8}$$

$$P_2^{(4)} - \tfrac{1}{2} \mathrm{Lie}_X^2\, P_2^{(0)} = \mathrm{Lie}_Y\, P_2^{(0)}, \tag{7.9}$$

where X is the vector field (7.2).

Proof. The reduction of the fourth-order deformation to the form $P_1^{(0)} - \lambda P_2^{(0)}$ can be achieved in two steps.

In the first step one kills the second order part of the deformation $P_2^{(0)}$ with the quasi-Miura transformation generated by the vector field (7.2):

$$P_1^{(0)} - \lambda (P_2^{(0)} + \epsilon^2 P_2^{(2)} + \epsilon^4 P_2^{(4)} + \mathrm{O}(\epsilon^5)) \to P_1^{(0)} - \lambda (P_2^{(0)} + \epsilon^4 (P_2^{(4)} - \tfrac{1}{2} \mathrm{Lie}_X^2\, P_2^{(0)} + \mathrm{O}(\epsilon^5))).$$

In the second step one kills the fourth order part of the deformation $P_2^{(4)} - 1/2\mathrm{Lie}_X^2\, P_2^{(0)}$ with the quasi-Miura transformation generated by the vector field Y. □

Theorem 8. *All fourth order deformations are quasi-trivial.*

To prove the theorem we need the following lemma.

Lemma 2. *If* $X_2^{(4)} = d_1\tilde{A} + d_2\tilde{B} - 1/2[d_1B, d_2B]$, *where*

$$B = \int_{S^1} b(u)\frac{u_{xx}}{u_x}\,\mathrm{d}x$$

and $b(u)$ *is given by the expression (7.7), then the fourth-order deformation*

$$P_1^{(0)} - \lambda(P_2^{(0)} + \epsilon^4(P_2^{(4)} - \tfrac{1}{2}\mathrm{Lie}_X^2\, P_2^{(0)}))$$

is trivial.

Proof. $P_2^{(4)} = d_1X_2^{(4)} = d_1d_2\tilde{B} - d_1 1/2[d_1B, d_2B]$.

By using the graded Jacobi identity we get

$$P_2^{(4)} - \tfrac{1}{2}\mathrm{Lie}_X^2\, P_2^{(0)} = \mathrm{Lie}_Y\, P_2^{(0)}, \tag{7.10}$$

where $X = -d_1B$ and $Y = -d_1\tilde{B}$. Moreover

$$\mathrm{Lie}_Y\, P_1^{(0)} = 0.$$ □

Proof of the theorem. In order to prove the theorem it is sufficient to show that if the coefficients s_i satisfy the conditions (6.21)–(6.23), then the vector field $X_2^{(4)} = s_0u_{xxxx} + s_1u_xu_{xxx} + s_2u_{xx}^2 + s_3u_x^2u_{xx} + s_4u_x^4$ can be written in the form

$$X_2^{(4)} = d_1\tilde{A} + d_2\tilde{B} - \tfrac{1}{2}[d_1B, d_2B].$$

We start calculating the term $-1/2[d_1B, d_2B]$. In this case the Schouten bracket coincides with the commutator of the vector fields

$$d_1B = 2\frac{\partial^2 b}{\partial u^2}u_{xx} - \frac{\partial b}{\partial u}\frac{u_{xx}^2}{u_x^2} + \frac{\partial b}{\partial u}\frac{u_{xxx}}{u_x} + \frac{\partial^3 b}{\partial u^3}u_x^2$$

and

$$d_2B = ud_1B + \frac{1}{2}\left(\frac{\partial b}{\partial u}u_{xx} + \frac{\partial^2 b}{\partial u^2}u_x^2\right).$$

The result is

$$
\begin{aligned}
-\frac{1}{2}[d_1B, d_2B]^x &= -\frac{1}{2}\sum_s\left(\partial_x^s(d_1B)\frac{\partial d_2B}{\partial u^{(s)}(x)} - \partial_x^s(d_2B)\frac{\partial d_1B}{\partial u^{(s)}(x)}\right)\\
&= u_{xxxx}\left(\frac{33}{4}\frac{\partial b}{\partial u}\frac{\partial^2 b}{\partial u^2}\right) + u_x u_{xxx}\left(\frac{27}{4}\left(\frac{\partial^2 b}{\partial u^2}\right)^2 + \frac{45}{4}\frac{\partial b}{\partial u}\frac{\partial^3 b}{\partial u^3}\right)\\
&\quad + u_{xx}^2\left(-\frac{3}{4}\left(\frac{\partial^2 b}{\partial u^2}\right)^2 + \frac{15}{4}\frac{\partial b}{\partial u}\frac{\partial^3 b}{\partial u^3}\right) + u_x^2 u_{xx}\left(\frac{45}{4}\frac{\partial^2 b}{\partial u^2}\frac{\partial^3 b}{\partial u^3}\right.\\
&\quad \left. + \frac{21}{2}\frac{\partial b}{\partial u}\frac{\partial^4 b}{\partial u^4}\right) + u_x^4\left(\frac{9}{4}\frac{\partial^2 b}{\partial u^2}\frac{\partial^4 b}{\partial u^3} + \frac{3}{2}\frac{\partial^5 b}{\partial u^5}\frac{\partial b}{\partial u} + \frac{3}{4}\left(\frac{\partial^3 b}{\partial u^3}\right)^2\right)\\
&\quad + \frac{9}{2}\frac{u_{xxx}^2}{u_x^2}\left(\frac{\partial b}{\partial u}\right)^2 - 9\frac{u_{xx}^4}{u_x^4}\left(\frac{\partial b}{\partial u}\right)^2 + 2\frac{u_{xxxxx}}{u_x}\left(\frac{\partial b}{\partial u}\right)^2\\
&\quad - \frac{u_{xx}u_{xxx}}{u_x}\left(\frac{27}{2}\frac{\partial b}{\partial u}\frac{\partial^2 b}{\partial u^2}\right) + \frac{u_{xx}^3}{u_x^2}\left(\frac{33}{4}\frac{\partial b}{\partial u}\frac{\partial^2 b}{\partial u^2}\right)\\
&\quad + 18\frac{u_{xx}^2 u_{xxx}}{u_x^3}\left(\frac{\partial b}{\partial u}\right)^2 - 6\frac{u_{xx}u_{xxxx}}{u_x^2}\left(\frac{\partial b}{\partial u}\right)^2.
\end{aligned}
$$

The problem now is to guess the form of the local functionals $\tilde{A}$ and $\tilde{B}$. We know that the degree of $\tilde{A}$ and $\tilde{B}$ is equal to 3 but all ratios of differential polynomials of the form

$$\frac{P(u, u_x, u_{xx}, \dots)}{Q(u, u_x, u_{xx}, \dots)},$$

where $\deg(P) - \deg(Q) = 3$, are allowed.

The form of the coefficients in (7.11) suggests that $\tilde{A}$ and $\tilde{B}$ contain only terms of the form

$$\frac{P(u, u_x, u_{xx}\cdots)}{u_x^i}$$

for some integer i.

Let us try with the local functionals:

$$
\begin{aligned}
\tilde{A} = \int_{s^1}\left(h_1(u)\frac{u_{xx}^2}{u_x} + h_2 2(u)\frac{u_{xx}^3}{u_x^3} + h_3(u)\frac{u_{xx}u_{xxx}}{u_x^2} + h_4(u)\frac{u_{xxxx}}{u_x} + h_5(u)\frac{u_{xx}^2 u_{xxx}}{u_x^4}\right.\\
+ h_6(u)\frac{u_{xx}^4}{u_x^5} + h_7(u)\frac{u_{xx}u_{xxxx}}{u_x^3} + h_8(u)\frac{u_{xxx}^2}{u_x^3} + h_9(u)\frac{u_{xxxxx}}{u_x^2}\\
\left. + h_{10}(u)\frac{u_{xxx}u_{xxxx}}{u_x^4} + h_{11}(u)\frac{u_{xxxx}^2}{u_x^5}\right)\mathrm{d}x,
\end{aligned}
$$

$$\tilde{B} = \int_{s^1} \left(k_1(u)\frac{u_{xx}^2}{u_x} + k_2(u)\frac{u_{xx}^3}{u_x^3} + k_3(u)\frac{u_{xx}u_{xxx}}{u_x^2} + k_4(u)\frac{u_{xxxx}}{u_x} + k_5(u)\frac{u_{xx}^2 u_{xxx}}{u_x^4} \right.$$
$$+ k_6(u)\frac{u_{xx}^4}{u_x^5} + k_7(u)\frac{u_{xx}u_{xxxx}}{u_x^3} + k_8(u)\frac{u_{xxx}^2}{u_x^3} + k_9(u)\frac{u_{xxxxx}}{u_x^2}$$
$$\left. + k_{10}(u)\frac{u_{xxx}u_{xxxx}}{u_x^4} + k_{11}(u)\frac{u_{xxxx}^2}{u_x^5} \right) \mathrm{d}x.$$

In order to determine the exact form of the coefficients h_i and k_i one has just to compare $d_1\tilde{a} + d_2\tilde{b}$ with $1/2[d_1 B, d_2 B]$.

The first step is to kill the coefficients of rational terms in $d_1\tilde{a} + d_2\tilde{b}$ that do not appear in $-1/2[d_1 B, d_2 B]$ and the second step is to eliminate the remaining rational terms.

The calculations are very long and we give here only the final result:

$$h_i(u) = -uk_i(u),\ i = 1, \dots, 11, \qquad k_3(u) = -\frac{1}{7}\frac{\partial k_8(u)}{\partial u} - \frac{15}{14}k_2(u) - k_4(u),$$
$$k_1(u) = -\frac{1}{28}\frac{\partial^2 k_8(u)}{\partial u^2} - \frac{15}{56}\frac{\partial k_2(u)}{\partial u}, \qquad k_4(u) = 0, \qquad k_5(u) = -6k_8(u),$$
$$k_6(u) = 4k_8(u), \qquad k_7(u) = k_8(u), \qquad k_9(u) = k_{10}(u) = k_{11}(u) = 0,$$
$$k_2(u) + 2\frac{\partial k_8}{\partial u} = \frac{7}{5}\left(\frac{\partial b}{\partial u}\right)^2.$$

In fact there is a certain freedom in the choice of coefficients h_i and k_i; for example it is not necessary to put $k_4(u)$ equal to 0.

With this choice and taking into account (7.7), by using formula (2.18) we get

$$-\frac{1}{2}[d_1 B, d_2 B] + d_1\tilde{A} + d_2\tilde{B}$$
$$= -2s\frac{\partial s}{\partial u}u_{xxxx} + \left(-\frac{7}{3}s\frac{\partial^2 s}{\partial u^2} - \frac{13}{3}\left(\frac{\partial s}{\partial u}\right)^2\right)u_x u_{xxx} + \left(-\frac{1}{3}s\frac{\partial^2 s}{\partial u^2} - \frac{7}{3}\left(\frac{\partial s}{\partial u}\right)^2\right)u_{xx}^2$$
$$+ \left(\frac{5}{3}s\frac{\partial^3 s}{\partial u^3} - 4\frac{\partial s}{\partial u}\frac{\partial^2 s}{\partial u^2}\right)u_x^2 u_{xx} + \left(\frac{1}{3}\left(\frac{\partial^2 s}{\partial u^2}\right)^2 + \frac{\partial s}{\partial u}\frac{\partial^3 s}{\partial u^3} + \frac{2}{3}s\frac{\partial^4 s}{\partial u^4}\right)u_x^4$$
$$= s'_0 u_{xxxx} + s'_1 u_x u_{xxx} + s'_2 u_{xx}^2 + s'_3 u_x^2 u_{xx} + s'_4 u_x^4.$$

First of all we observe that s'_0 is equal to s_0. Moreover the coefficients s'_i satisfy Eqs. (6.22) and (6.23):

$$s'_1 - s'_2 = \frac{\partial}{\partial u}\left(-2s\frac{\partial s}{\partial u}\right),$$
$$3s'_4 - \frac{\partial s'_3}{\partial u} + \frac{\partial^2 s'_2}{\partial u^2} = \left(\frac{\partial^2 s}{\partial u^2}\right)^2 + 3\frac{\partial s}{\partial u}\frac{\partial^2 s}{\partial u^2} + 2s\frac{\partial^4 s}{\partial u^4} - \frac{5}{3}s\frac{\partial^4 s}{\partial u^4} + \frac{7}{3}\frac{\partial s}{\partial u}\frac{\partial^3 s}{\partial u^3}$$
$$+ 4\left(\frac{\partial^2 s}{\partial u^2}\right)^2 - \frac{16}{3}\frac{\partial s}{\partial u}\frac{\partial^3 s}{\partial u^3} - 5\left(\frac{\partial^2 s}{\partial u^2}\right)^2 - \frac{1}{3}s\frac{\partial^4 s}{\partial u^4} = 0.$$

Then the vector field

$$X - (-\tfrac{1}{2}[d_1 B, d_2 B] + d_1\tilde{A} + d_2\tilde{B}) = (s_0 - s_0')u_{xxxx} + (s_1 - s_1')u_x u_{xxx} + (s_2 - s_2')u_{xx}^2 + (s_3 - s_3')u_x^2 u_{xx} + (s_4 - s_4')u_x^4$$

is trivial. Indeed

$$s_0 - s_0' = 0.$$

Moreover the equations

$$s_1 - s_2 = \frac{\partial s_0}{\partial u}, \qquad s_1' - s_2' = \frac{\partial s_0}{\partial u}$$

imply

$$s_1 - s_1' = s_2 - s_2'.$$

Finally

$$3(s_4 - s_4') - \frac{\partial}{\partial u}(s_3 - s_3') + \frac{\partial^2}{\partial u^2}(s_2 - s_2') = 3s_4 - \frac{\partial}{\partial u}s_3 + \frac{\partial^2}{\partial u^2}s_2 - \left(3s_4' - \frac{\partial}{\partial u}s_3' + \frac{\partial^2}{\partial u^2}s_2'\right) = 0.$$

This means that

$$X - (-\tfrac{1}{2}[d_1 B, d_2 B] + d_1\tilde{A} + d_2\tilde{B}) = d_1 A' + d_2 B',$$

i.e.:

$$X = -\tfrac{1}{2}[d_1 B, d_2 B] + d_1(\tilde{A} + A') + d_2(\tilde{B} + B'). \qquad \square$$

7.3. *Example*

Dubrovin and Zhang [7] show that the genus expansion of the topological gravity coincides with the quasi-trivial transformation for KdV hierarchy. More precisely under the action of the transformation

$$u \to u + \frac{\epsilon^2}{24}(\log u_x)_{xx} + \epsilon^4\left(\frac{u^{(4)}}{1152u'^2} - \frac{7u''u'''}{1920u'^3} + \frac{u''^3}{360u'^4}\right)_{xx} + \mathrm{O}(\epsilon^6) \tag{7.11}$$

the bi-Hamiltonian structure of the dispersionless KdV hierarchy becomes the Magri bi-Hamiltonian structure.

Starting from the theorem proved in the last section it is possible to construct the transformation (7.11) up to the fourth order. In fact we have seen that in order to reduce a fourth-order deformation to the form $P_1^{(0)} - \lambda P_2^{(0)}$ two quasi-Miura transformations and, eventually, one Miura transformation have to be performed. The quasi-Miura transformation is given by the formula

$$u \to \bar{u} = \exp(\epsilon^4 Y)\exp(\epsilon^2 X)u, \tag{7.12}$$

where $X = -d_1 B$ and $Y = -d_1 \tilde{B}$ (see the previous section).

In the KdV case $s =$ constant $= c$. This implies

$$B = -\frac{2}{3} c \int_{S^1} \left(u \frac{u_{xx}}{u_x} \right) \mathrm{d}x. \tag{7.13}$$

It follows easily that

$$X = \sum_{k=0} X^{(k)} \frac{\partial}{\partial u^{(k)}}, \tag{7.14}$$

where

$$X^{(0)} = k \left(\frac{u_{xxx}}{u_x} - \frac{u_{xx}^2}{u_x^2} \right). \tag{7.15}$$

$X^{(k)}$ denotes the kth derivative of $X^{(0)}$ and $k = -2/3c$

The choice of $\tilde{B}$ is not unique. One of the possible choices is the following one:

$$\tilde{B} = k^2 \left(\frac{7u_{xx}^3}{5u_x^3} - \frac{3u_{xx}u_{xxx}}{2u_x^2} \right). \tag{7.16}$$

This implies

$$Y = \sum_{k=0} Y^{(k)} \frac{\partial}{\partial u^k}, \tag{7.17}$$

where

$$Y^{(0)} = k^2 \left(-\frac{9u_{xxx}u_{xxxx}}{5u_x^3} - \frac{3u_{xx}u_{xxxxx}}{3u_x^3} + \frac{27u_{xx}^2 u_{xxxx}}{5u_x^4} + \frac{9u_{xx}u_{xxx}^2}{u_x^4} - \frac{24u_{xx}^3 u_{xxx}}{u_x^5} + \frac{12u_{xx}^5}{u_x^6} \right). \tag{7.18}$$

It is easy to check that the transformation

$$u \to \exp(\epsilon^4 Y) \exp(\epsilon^2 X) u = (\mathrm{Id} + \epsilon^2 X + \epsilon^4 (Y + \tfrac{1}{2} X^2) + \mathrm{O}(\epsilon^6)) u \tag{7.19}$$

coincides up to the fourth order with the transformation (7.11). No additional Miura transformation is necessary in the KdV case.

8. Conclusions

In this paper we have studied the problem of classification of deformations of bi-Hamiltonian structure of hydrodynamic type.

The main result of the paper is that, up to the fourth order, any deformation can be reduced to the form (3.15) depending only on a functional parameter $s(u)$.

These deformations give rise to an infinite hierarchy of "almost-commuting" Hamiltonian equations.

In this paper we started studying numerically one of the equations of the deformed hierarchy corresponding to second-order deformations.

The results obtained indicate the existence of the analogue of two-soliton solutions at least for small times and for small amplitudes, but a deeper analysis is still necessary.

According to [2] a classification problem can be thought as a decomposition of a space of objects in equivalence classes and an object is called stable if a sufficiently small "neighborhood" of this object contains only objects of the same class. In our case the objects are deformations; two deformations are equivalent if and only if they can be reduced to the same form by the action of Miura group. Moreover we have seen that one of the consequence of classification theorem is that deformations of the Magri bi-Hamiltonian structure are trivial up to fourth order. This fact suggests that Magri bi-Hamiltonian structure is a stable object.

It would be interesting to investigate if this situation is more general, i.e. if completely integrable systems of bi-Hamiltonian type are the stable objects of a corresponding classification problem. For example it is possible to apply the same techniques used in this paper to the "symplectic" case where the leading term in the parameter ϵ has the form

$$P_{1,2} = h^{ij}_{1,2}\delta(x-y) \tag{8.1}$$

with

$$h_1^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}$$

and

$$h_2^{ij} = \begin{pmatrix} 0 & u_1 & 0 & 0 & \cdots & 0 & 0 \\ -u_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & u_2 & \cdots & 0 & 0 \\ 0 & 0 & -u_2 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & u_n \\ 0 & 0 & 0 & 0 & \cdots & -u_n & 0 \end{pmatrix}.$$

Also in this case the the cohomology groups H^1 and H^2 associated to the differentials d_{P_1} and d_{P_2} are trivial (see [7]) and then the non-trivial deformations are classified by the group

$$\frac{\ker(d_1 d_2)}{(\mathrm{Im}(d_1) + \mathrm{Im}(d_2))}.$$

Finally we observe that the result about quasi-triviality suggests that this property is a consequence of the definition of deformation and it is not an additional constraint.

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